On the Fundamental Transverse Vibration Frequency of a Free-Free Thin Beam With Identical End Masses

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Current research in vibration-based energy harvesting and in microelectromechanical system technology has focused renewed attention on the vibration of beams with end masses. This paper shows that the commonly accepted and frequently quoted fundamental natural frequency formula for a beam with identical end masses is incorrect. It is also shown that the higher mode frequency expressions suggested in the referred work (Haener, J., 1958, "Formulas for the Frequencies Including Higher Frequencies of Uniform Cantilever and Free-Free Beams With Additional Masses at the Ends," ASME J. Appl. Mech. **25***, pp. 412) are also incorrect. The correct characteristic (frequency) equation is derived and nondimensional comparisons are made between the results of the previously published formula and the corrected formulation using Euler–Bernoulli beam assumptions. The previous formula is shown to be accurate only for the extreme case of very large end mass to beam mass ratios. Curve fitting is used to report alternative first order and second order polynomial ratio expressions for the first natural frequency, as well as for the frequencies of some higher modes.* [DOI: 10.1115/1.2776341]

1 Introduction

The recent interest in energy harvesting [1–3] using various cantilever beams covered with piezoceramic material has renewed interest in simple single degree-of-freedom formulas for beams with a single end mass. Much of the modeling work in energy harvesting has focused on quoting simple formulas for the frequency of a beam with an end mass. For representing the dynamics of some particular energy harvesters, it was noted that a free-free beam with end masses may be useful. In deriving an analytical formulation for energy harvesting using a beam with two end masses, the previously published beam frequency equations stemming from the work of Haener [4] decades ago and reported in many handbooks and tabulations [5,6] was examined. Unfortunately, the early natural frequency formulas (both for the fundamental natural frequency and for the higher mode frequencies) proposed by Haener [4] are incorrect, yet they still persist. The goal of this paper is to provide the correct calculation of the natural frequencies of thin beams with identical end masses.

2 On the Existing Fundamental Natural Frequency Formula

For decades, the following expression suggested by Haener [4] has appeared in various structural dynamics reference books [5,6] for representing the fundamental transverse vibration frequency of a free-free slender (Euler–Bernoulli) beam with identical masses at both ends:

$$f_1 = \frac{\pi}{2}\sqrt{\left[1 + \frac{5.45}{1 - 77.4(M/m_b)^2}\right]\frac{EI}{m_b L^3}} \tag{1}$$

where E is Young's modulus, I is the cross-sectional area moment of inertia, L is the length, m_b is the total mass of the beam, and M is the mass rigidly attached to each end of the beam. This form of Eq. (1) gives the natural frequency in hertz and it can be converted to rad/s according to $\omega_1 = 2\pi f_1$. After rearranging Eq. (1) by separating the dimensionless frequency term from the material and geometric parameters, one can obtain the following equation, which gives the resulting natural frequency in rad/s:

$$\omega_1 = \pi^2\sqrt{1 + \frac{5.45}{1 - 77.4(M/m_b)^2}}\sqrt{\frac{EI}{m_b L^3}} \tag{2}$$

It is clear from Eq. (2) that it may yield imaginary numbers and may even tend to infinity for certain values of M/m_b (end mass to beam mass ratio). For this fundamental natural frequency expression to yield a finite positive real value, the following inequality must be satisfied:

$$1 + \frac{5.45}{1 - 77.4(M/m_b)^2} > 0 \tag{3}$$

which can be reduced to the following acceptable ranges of M/m_b:

$$0 \leqslant \frac{M}{m_b} < 0.1137 \quad \frac{M}{m_b} > 0.2887 \tag{4}$$

and for the following range of M/m_b, the expression suggested by Haener [4] results in imaginary numbers, which has no physical meaning:

$$0.1137 < \frac{M}{m_b} < 0.2887 \tag{5}$$

The natural frequency of the rth mode of an Euler–Bernoulli beam can be expressed as [5]

$$\omega_r = \lambda_r^2\sqrt{\frac{EI}{m_b L^3}} \tag{6}$$

where λ_r is well known as the dimensionless frequency parameter of the rth mode [5]. Therefore, according to Eq. (2), the dimensionless frequency parameter of the first elastic mode suggested by Haener [4] is

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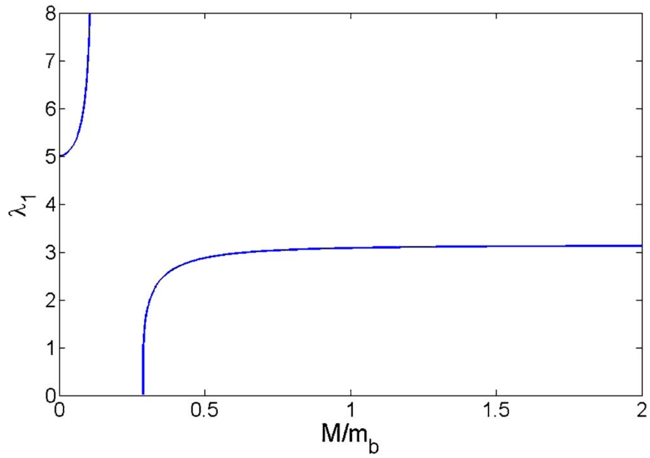

Fig. 1 Dimensionless frequency parameter of the first vibration mode calculated from the relation suggested by Haener [4] versus end mass to beam mass ratio

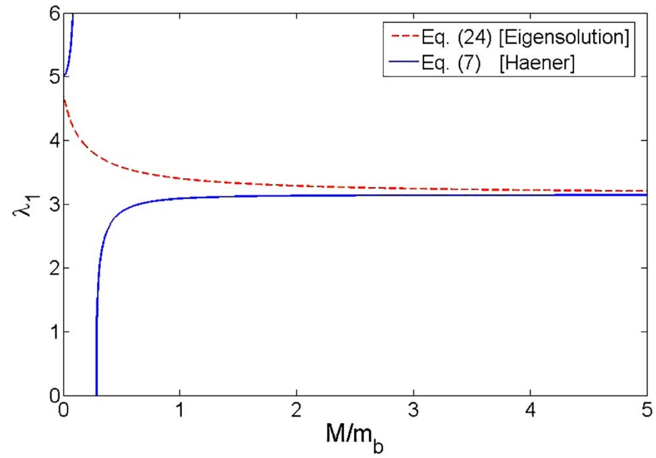

Fig. 2 Dimensionless frequency parameter of the first mode calculated from the eigensolution and from the relation suggested by Haener [4] versus end mass to beam mass ratio

$$\lambda_1 = \pi\left[1 + \frac{5.45}{1 - 77.4(M/m_b)^2}\right]^{1/4} \tag{7}$$

When λ_1 is plotted against M/m_b, the invalid end mass to beam mass ratio region given by Eq. (5) can be seen clearly (Fig. 1). The question arising at this point is as follows: Does Eq. (1) yield correct results for the M/m_b regions given by Eq. (4) where it gives positive real numbers? Probably the simplest case one can check is $M=0$ (and therefore $M/m_b=0$), which is the case when there are no end masses. In this particular case, the beam reduces to a uniform free-free Euler–Bernoulli beam without end masses, whose dimensionless frequency parameter for the first mode[2] is $\lambda_1=4.7300$. However, when $M=0$ is used in Eq. (1), one obtains

$$\lambda_1 = \pi[1 + 5.45]^{1/4} = 5.0066 \tag{8}$$

which has 5.8% relative error. It should be noted from Eq. (6) that the natural frequency ω_1 is proportional to λ_1^2 and, therefore, the relative error in the resulting natural frequency predicted by using Eq. (1) for the particular case of $M=0$ is about 12%.

In the original paper [4], without any explanation, the term $(M/m_b)^2>0.08$ appears next to the fundamental natural frequency expression given by Eq. (1), which is not mentioned in the books of Blevins [5] and Pilkey [6]. This range yields $M/m_b>0.2828$, and it is roughly one of the limitations we obtained in Eq. (4). However, it is important to notice from Fig. 1 that this limitation is just for Eq. (1) to give real valued finite results, not necessarily correct fundamental natural frequency predictions. Moreover, Haener [4] claims that "the value of the natural frequency obtained in Eq. (1) is about 1% too large for $M/m_b>0.02$ and is good to three decimals for $0.1553<M/m_b<1$," which means that he was not aware of the fact that his equation gives imaginary numbers for $0.1137<M/m_b<0.2887$. As will be shown, Eq. (1) is incorrect even with this restriction since the characteristic equation given in the original paper [4] is incorrect.

In the following sections, we first derive the characteristic (frequency) equation for the transverse vibrations of a free-free Euler–Bernoulli beam with identical masses at both ends and show that it is different from the one presented by Haener [4]. The difference and the relative error are then highlighted by comparing the resulting dimensionless frequency parameter predictions graphically against mass ratio. It is also shown that the higher mode frequency relations suggested by Haener [4] are also incorrect. Finally, the correct dimensionless frequency parameters for the first five modes are plotted against end mass to beam mass ratio and simple relations obtained by curve fitting are presented for the first five natural frequencies.

[2]It is the first nonzero root of the transcendental equation $1-\cos\lambda\cosh\lambda=0$ (see, for instance, Blevins [5]).

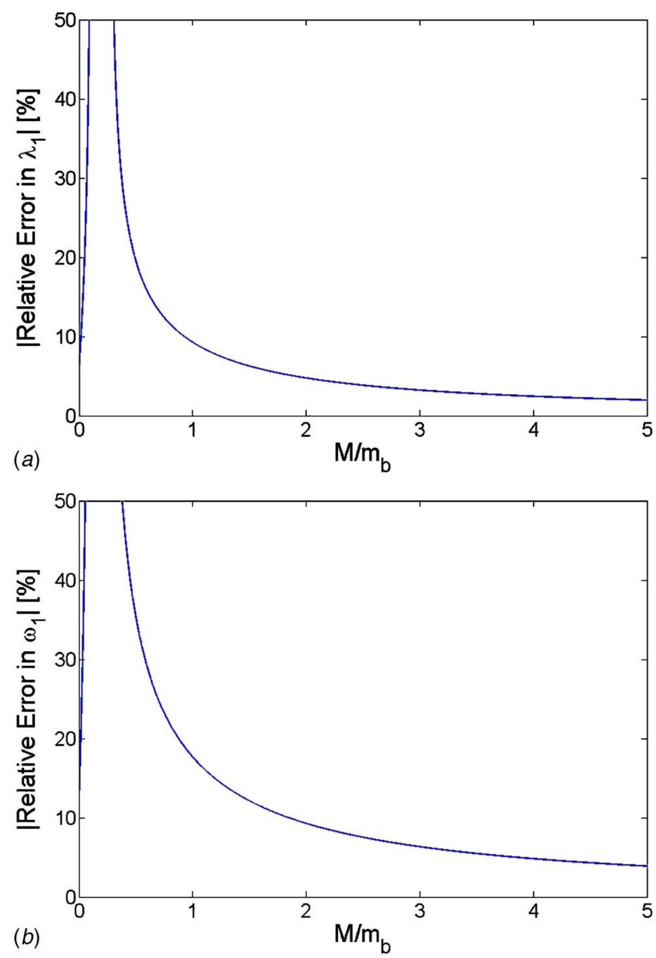

Fig. 3 Relative percentage error due to using the relation suggested by Haener [4] (*a*) in the dimensionless frequency parameter and (*b*) in the fundamental natural frequency

3 Mathematical Formulation

The equation of motion of an undamped Euler–Bernoulli beam with uniform cross section can be written as follows and its derivation is available in numerous vibration texts [7]:

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} + \frac{m_b}{L}\frac{\partial^2 w(x,t)}{\partial t^2} = 0 \tag{9}$$

where $w(x,t)$ is the displacement in the transverse direction and x denotes the axial position along the beam. According to the sign convention used by Timoshenko and Young [8], the bending moment and shear force can be expressed as

$$M(x,t) = EI\frac{\partial^2 w(x,t)}{\partial x^2} \quad S(x,t) = EI\frac{\partial^3 w(x,t)}{\partial x^3} \tag{10}$$

The moment equilibrium at the boundaries gives

$$EI\frac{\partial^2 w(x,t)}{\partial x^2}\bigg|_{x=0} = 0 \quad EI\frac{\partial^2 w(x,t)}{\partial x^2}\bigg|_{x=L} = 0 \tag{11}$$

and one can obtain the following equations from the force equilibrium at the boundaries:

$$M\frac{\partial^2 w(x,t)}{\partial t^2}\bigg|_{x=0} + EI\frac{\partial^3 w(x,t)}{\partial x^3}\bigg|_{x=0} = 0$$
$$M\frac{\partial^2 w(x,t)}{\partial t^2}\bigg|_{x=L} - EI\frac{\partial^3 w(x,t)}{\partial x^3}\bigg|_{x=L} = 0 \tag{12}$$

It should be noted that, while writing Eq. (11), the rotary inertias of the end masses are neglected to be consistent with Haener [4].

Using separation of variables for the solution of Eq. (9) by setting $w(x,t)=\bar{w}(x)f(t)$ yields the following ordinary differential equations in spatial domain and time domain, respectively:

$$\frac{d^4\bar{w}(x)}{dx^4} - \omega^2\frac{m_b}{EIL}\bar{w}(x) = 0 \tag{13}$$

$$\frac{d^2 f(t)}{dt^2} + \omega^2 f(t) = 0 \tag{14}$$

Employing the following relation between the frequency ω and the dimensionless parameter λ:

$$\lambda^4 = \omega^2\frac{m_b L^3}{EI} \tag{15}$$

the solutions of Eqs. (13) and (14) can be expressed as

$$\bar{w}(x) = A\cos\frac{\lambda x}{L} + B\cosh\frac{\lambda x}{L} + C\sin\frac{\lambda x}{L} + D\sinh\frac{\lambda x}{L} \tag{16}$$

$$f(t) = E\cos\omega t + F\sin\omega t \tag{17}$$

where A, B, C, D, E, and F are constants. After separating the variables, the boundary conditions given by Eqs. (11) and (12) are reduced to

$$\frac{d^2\bar{w}(x)}{dx^2}\bigg|_{x=0} = 0 \quad \frac{d^2\bar{w}(x)}{dx^2}\bigg|_{x=L} = 0 \tag{18}$$

$$\frac{\lambda^4}{L^3}\frac{M}{m_b}\bar{w}(0) - \frac{d^3\bar{w}(x)}{dx^3}\bigg|_{x=0} = 0 \quad \frac{\lambda^4}{L^3}\frac{M}{m_b}\bar{w}(L) + \frac{d^3\bar{w}(x)}{dx^3}\bigg|_{x=L} = 0 \tag{19}$$

When Eq. (16) is used in Eqs. (18) and (19), one obtains

$$-A + B = 0 \quad -A\cos\lambda + B\cosh\lambda - C\sin\lambda + D\sinh\lambda = 0 \tag{20}$$

$$-A\frac{M}{m_b}\lambda - B\frac{M}{m_b}\lambda - C + D = 0 \tag{21}$$

$$A\left(\frac{M}{m_b}\lambda\cos\lambda + \sin\lambda\right) + B\left(\frac{M}{m_b}\lambda\cosh\lambda + \sinh\lambda\right) + C\left(\frac{M}{m_b}\lambda\sin\lambda - \cos\lambda\right) + D\left(\frac{M}{m_b}\lambda\sinh\lambda + \cosh\lambda\right) = 0 \tag{22}$$

Then, Eqs. (20)–(22) can be expressed in the matrix form as

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -\mathrm{c}\,\lambda & \mathrm{ch}\,\lambda & -\mathrm{s}\,\lambda & \mathrm{sh}\,\lambda \\ -(M/m_b)\lambda & -(M/m_b)\lambda & -1 & 1 \\ (M/m_b)\lambda\,\mathrm{c}\,\lambda + \mathrm{s}\,\lambda & (M/m_b)\lambda\,\mathrm{ch}\,\lambda + \mathrm{sh}\,\lambda & (M/m_b)\lambda\,\mathrm{s}\,\lambda - \mathrm{c}\,\lambda & (M/m_b)\lambda\,\mathrm{sh}\,\lambda + \mathrm{ch}\,\lambda \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{23}$$

where c, s, ch, and sh stand for cos, sin, cosh, and sinh, respectively. Note that the variable elements of the above coefficient matrix are functions of the dimensionless parameters λ and M/m_b only. For nontrivial solutions of Eq. (23), the coefficient matrix should be singular and, therefore, its determinant should vanish, yielding

$$1 - \cos\lambda\cosh\lambda + 2\frac{M}{m_b}\lambda(\sin\lambda\cosh\lambda - \sinh\lambda\cos\lambda) + 2\left(\frac{M}{m_b}\lambda\right)^2\sin\lambda\sinh\lambda = 0 \tag{24}$$

At this point, we compare the above characteristic equation with the one obtained by Haener [4] for a "constant cross-section free-free beam with additional masses at both ends:"

$$1 - \cos\lambda\cosh\lambda - 2\left(\frac{M}{m_b}\lambda\right)^2\sin\lambda\sinh\lambda = 0 \tag{25}$$

As can be seen, the above characteristic equations given by Eqs. (24) and (25) are considerably different from each other. Not only one term appearing in Eq. (24) is missing in Eq. (25), but the sign of the last term in the latter equation is also incorrect. Comparison of the results of these equations and the relevant discussion are given in the following section.

4 Results and Discussion

It is worthwhile to mention that what we define as the first mode in this work is the first elastic mode of the structure, i.e., we are not referring to the translational and/or rotational rigid body modes of the structure that correspond to the eigenvalue $\lambda_0=0$ of

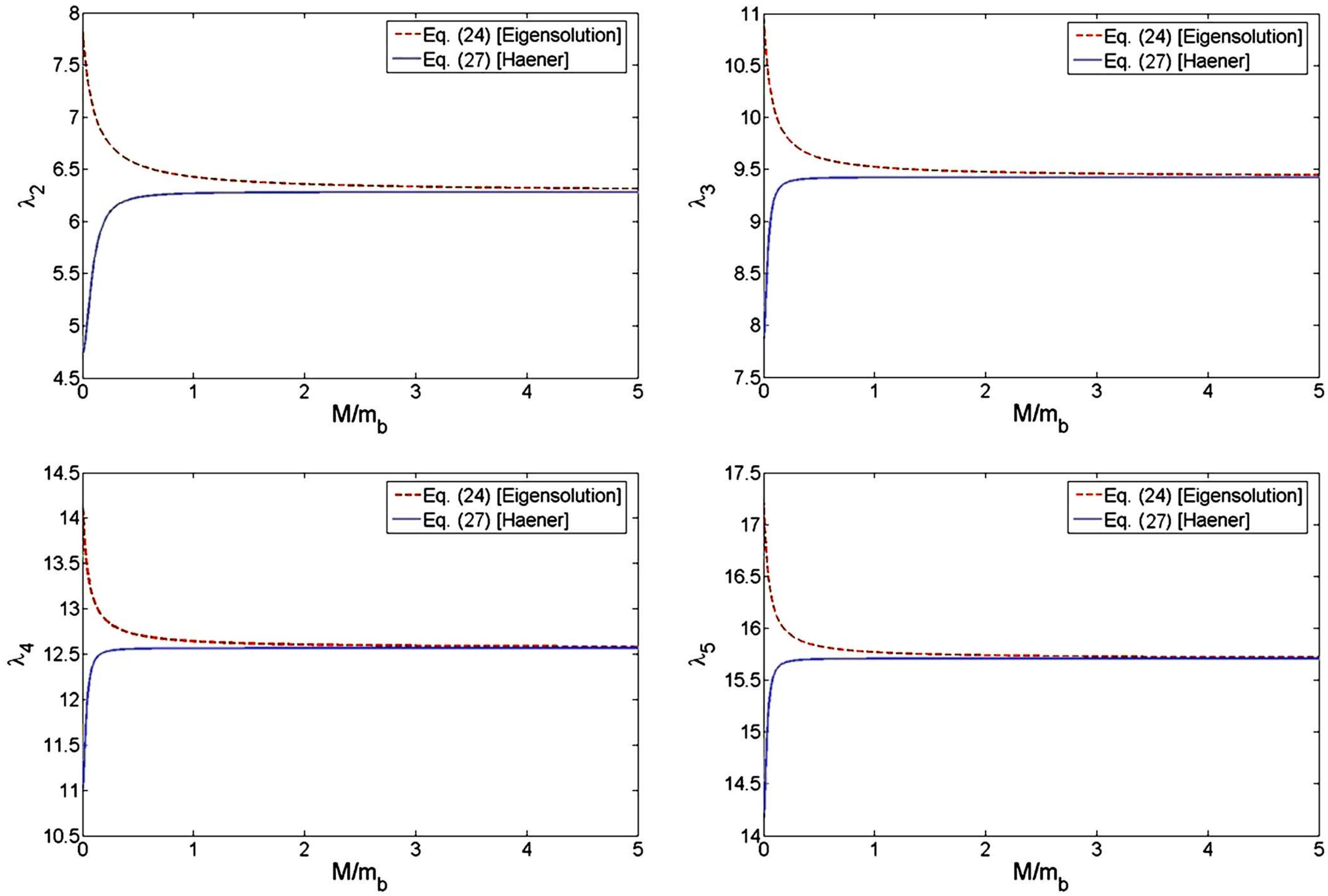

Fig. 4 Dimensionless frequency parameters of the higher modes calculated from the eigensolution and from the relations suggested by Haener [4] versus end mass to beam mass ratio

Eq. (24). Dimensionless frequency parameters obtained from the numerical solution of Eq. (24) for the fundamental vibration mode and the dimensionless frequency parameter expression (Eq. (7)) extracted from the relation suggested by Haener [4] are compared in Fig. 2 for the end mass to beam mass ratio range of $0 < M/m_b < 5$.

It can be seen from Fig. 2 that the resulting dimensionless frequency parameters calculated using Eq. (7) and those calculated using Eq. (24) can be said to be in agreement only for very high end mass to beam mass ratios. As mentioned previously, the relation suggested by Haener [4] is presented with $(M/m_b)^2 > 0.08$ in his original paper, which is identical to $M/m_b > 0.2828$. Although this region roughly covers the values[3] where Eq. (1) yields real numbers, the resulting dimensionless frequency parameters, and therefore the fundamental natural frequencies predicted by his relation in this range, are far from being accurate (Fig. 2). Note that, since the natural frequency is proportional to the square of the dimensionless frequency parameter according to Eq. (6), the error in the predicted natural frequency is much higher than that in the predicted frequency parameter. Figures 3(*a*) and 3(*b*) display the relative percentage errors in dimensionless frequency parameter and natural frequency due to using the relation suggested by Haener [4], respectively.

It should be mentioned that the higher mode frequency relations suggested by Haener [4] are also incorrect. In the relevant paper [4], natural frequency expressions for the second and the higher modes are given by the following equations:

$$\omega_2 = \pi^2 \sqrt{\left[16 - \frac{10.86}{77.7(M/m_b)^2 + 1}\right] \frac{EI}{m_b L^3}}$$

$$\omega_n = \pi^2 \sqrt{\left[n^4 - \frac{n^4 - (n-0.5)^4}{17.5n(M/m_b)^2 + 1}\right] \frac{EI}{m_b L^3}} \quad n > 2 \qquad (26)$$

For a nondimensional comparison, in the same manner (as we did for the fundamental natural frequency case), the dimensionless frequency parameters are extracted from the foregoing natural frequency relations suggested by Haener [4], as follows:

$$\lambda_2 = \pi \left[16 - \frac{10.86}{77.7(M/m_b)^2 + 1}\right]^{1/4}$$

$$\lambda_n = \pi \left[n^4 - \frac{n^4 - (n-0.5)^4}{17.5n(M/m_b)^2 + 1}\right]^{1/4} \quad n > 2 \qquad (27)$$

These expressions will now be compared with the higher roots of the characteristic equation (Eq. (24)), which we obtained as a function of end mass to beam mass ratio. Having shown the discrepancy in the dimensionless frequency parameter of the first mode extracted from the equation suggested by Haener [4] (Fig. 2), we now present the differences between the higher roots of Eq. (24) and the values obtained by Eq. (27) as a function of end mass to beam mass ratio in Fig. 4. As can be seen from Fig. 4, the predictions of Eq. (27) can be said to be reasonable only for very high M/m_b values. For relatively low M/m_b values, Eq. (27) yields totally incorrect results.

[3]See the exact ranges given by Eq. (4).

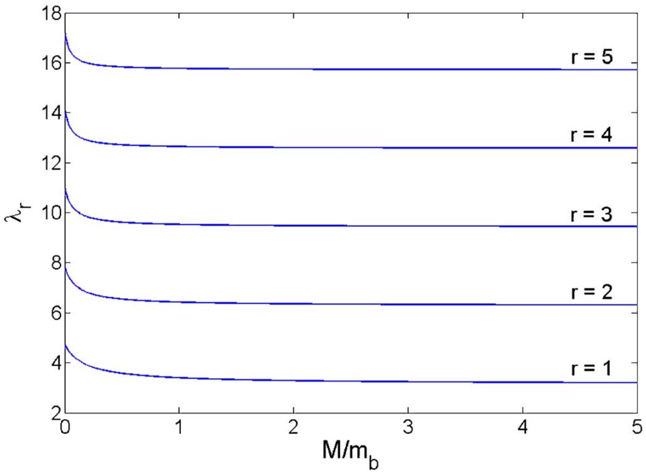

Fig. 5 Dimensionless frequency parameters versus end mass to beam mass ratio for the first five modes

It is obvious from Fig. 4 that, for the extreme case of $M/m_b=0$, the dimensionless frequency parameter predicted by Eq. (27) for the rth mode actually belongs to the $(r-1)$th mode. For instance, we previously discussed that, for the particular case of $M/m_b=0$, the free-free beam has no end masses and, therefore, the first dimensionless frequency parameter should come out to be $\lambda_1=4.7300$ [5], which is also what we predicted in Fig. 2 by using Eq. (24). However, when $M/m_b=0$ is used in Eq. (27), one obtains $\lambda_2=4.7300$, $\lambda_3=7.8532$ (whereas, actually, $\lambda_2=7.8532$), and so on, since the asymptotic behavior of Eq. (27) for relatively low M/m_b ratios are totally incorrect. Once again, we underline that the corresponding natural frequencies are proportional to the squares of the dimensionless frequency parameters and, therefore, the deviations in the natural frequencies predicted by the equations suggested by Haener [4] are much higher than those in the dimensionless frequency parameters.

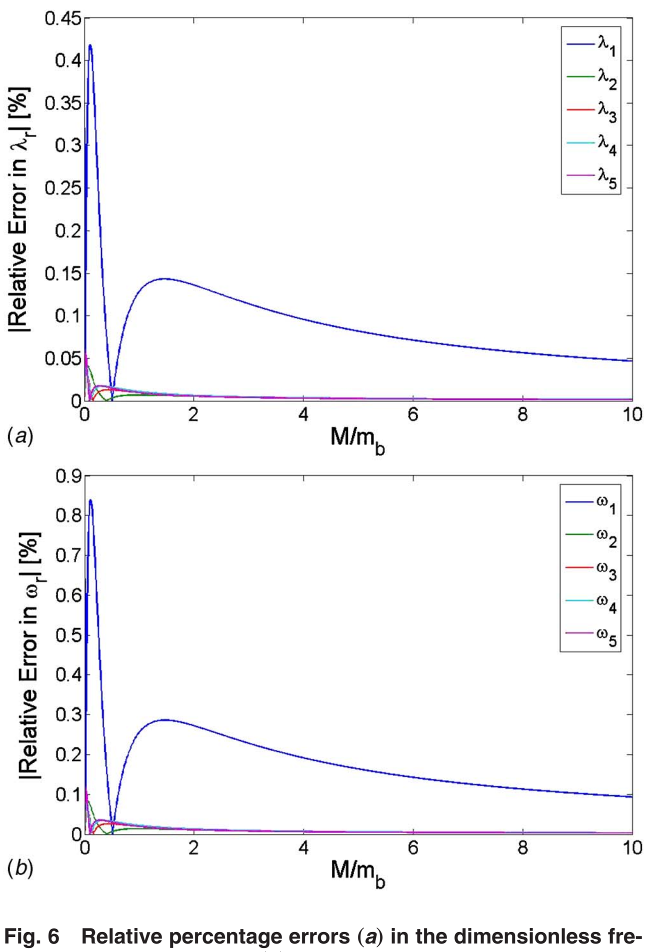

Fig. 6 Relative percentage errors (*a*) in the dimensionless frequency parameters and (*b*) in the natural frequencies due to using the first order polynomial ratios given by Eq. (30)

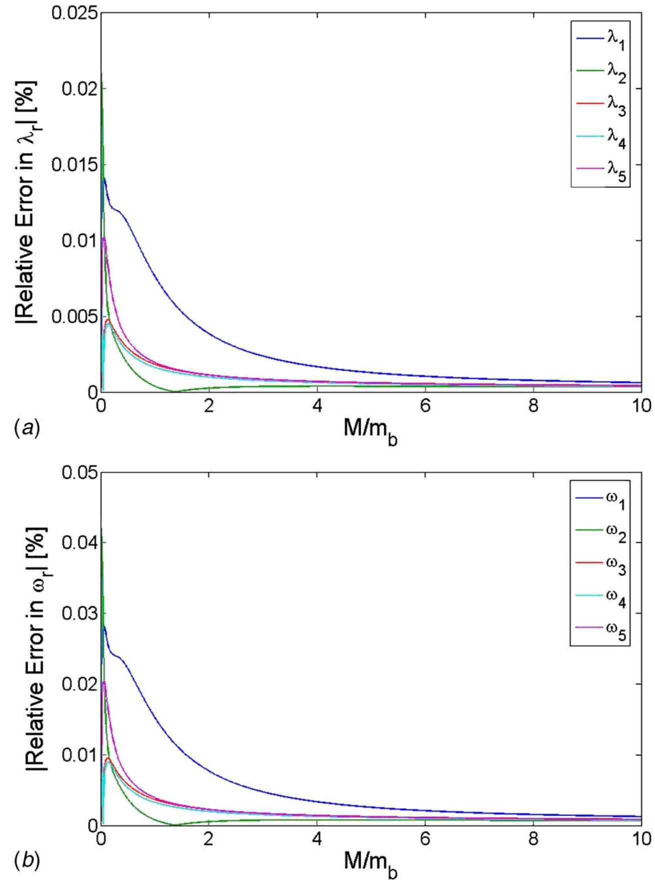

Fig. 7 Relative percentage errors (*a*) in the dimensionless frequency parameters and (*b*) in the natural frequencies due to using the second order polynomial ratios given by Eq. (31)

5 New Expressions for the First Five Natural Frequencies

Having completed our discussion related to Eq. (1) suggested by Haener [4] that has been used in different reference texts [5,6] for decades and also the discussion regarding the higher mode frequency equations suggested in the same work, we plot the correct dimensionless frequency parameter versus end mass to beam mass ratio plots for the first five modes together in Fig. 5. These dimensionless frequency parameters must be used in Eq. (6) in obtaining the corresponding natural frequencies. It should be noted from Eq. (24) and Fig. 5 that the λ values obtained for $M/m_b=0$ are the roots of the transcendental equation

$$1-\cos\lambda\cosh\lambda=0 \tag{28}$$

which is the characteristic equation of a free-free Euler–Bernoulli beam with no end masses [5]. For high M/m_b values, Eq. (24) reduces to

$$\sin\lambda=0 \tag{29}$$

since the last term in Eq. (24) becomes the dominant term as $M/m_b\to\infty$ and $\sinh\lambda\neq0$ for $\lambda\neq0$. Equation (29) is the characteristic equation of a simply supported uniform Euler–Bernoulli beam [5] and its roots are the integer multiples of π, which can also be seen from Fig. 5. This observation makes sense because in case of very large end masses, the transverse motions at the end points of the beam are restricted. However, since the rotary inertias of the end masses are neglected in the formulation, the rotational motions at these points are allowed (there is no resisting moment): hence, the boundary conditions for high M/m_b values become simply supported.

In order to represent the curves given in Fig. 5, the curve fitting toolbox of MATLAB® is used. For each mode, both first order and quadratic polynomial ratios are obtained with confidence bounds of 99.9%. The following expressions are the first order polynomial ratios for the dimensionless frequency parameters:

$$\lambda_1 = \frac{3.1416\alpha + 0.9056}{\alpha + 0.1923} \quad \lambda_2 = \frac{6.2832\alpha + 0.7908}{\alpha + 0.1007}$$

$$\lambda_3 = \frac{9.4248\alpha + 0.7484}{\alpha + 0.06801} \tag{30}$$

$$\lambda_4 = \frac{12.5664\alpha + 0.7268}{\alpha + 0.05136} \quad \lambda_5 = \frac{15.7080\alpha + 0.7141}{\alpha + 0.04129}$$

where $\alpha = M/m_b$. The errors in the dimensionless frequency parameters obtained using the above expressions are less than 0.45% for all values of end mass to beam mass ratio (M/m_b) and the error decreases as the ratio M/m_b increases (Fig. 6(*a*)). It should be noted that the error estimates are relative to the roots of the transcendental Eq. (24), which is based on Euler–Bernoulli beam assumptions. As can be seen in Fig. 6(*b*), the errors reflected to the natural frequencies due to using Eq. (30) are less than 0.9%.

For higher accuracy, alternatively, we also present the following quadratic polynomial ratio relations for the dimensionless frequency parameters:

$$\lambda_1 = \frac{3.1416\alpha^2 + 1.821\alpha + 0.2211}{\alpha^2 + 0.4783\alpha + 0.04674}$$

$$\lambda_2 = \frac{6.2832\alpha^2 + 2.408\alpha + 0.2003}{\alpha^2 + 0.3579\alpha + 0.02551}$$

$$\lambda_3 = \frac{9.4248\alpha^2 + 1.258\alpha + 0.04291}{\alpha^2 + 0.1222\alpha + 0.003903} \tag{31}$$

$$\lambda_4 = \frac{12.5664\alpha^2 + 1.304\alpha + 0.03672}{\alpha^2 + 0.09742\alpha + 0.002598}$$

$$\lambda_5 = \frac{15.7080\alpha^2 + 1.303\alpha + 0.03015}{\alpha^2 + 0.07888\alpha + 0.001745}$$

The relative errors due to using the above expressions instead of finding the roots of Eq. (24) for the first five modes are less than 0.025% for all values of $\alpha = M/m_b$. Once again, the error decreases with increasing M/m_b (Fig. 7(*a*)). According to Fig. 7(*b*), the errors in the resulting natural frequencies are less than 0.05% when the frequency parameter expressions given by Eqs. (31) are used in Eq. (6).

Figures 8(*a*) and 8(*b*), respectively, display the linear and the quadratic polynomial ratio curve fits given by Eqs. (30) and (31) along with the exact solution. There is no notable difference in the resulting graphs, yet the small difference in the errors of linear and quadratic fits can be seen in Figs. 6 and 7.

Either the first order polynomial ratios given by Eq. (30) or the quadratic polynomial ratios given by Eq. (31) can be used in Eq. (6) to obtain the first five natural frequencies with very good accuracy. Therefore, either of the following expressions can be considered as a candidate to take the place of the incorrect fundamental natural frequency equation suggested by Haener [4], which has been used in the books of Blevins [5] and Pilkey [6] for decades:

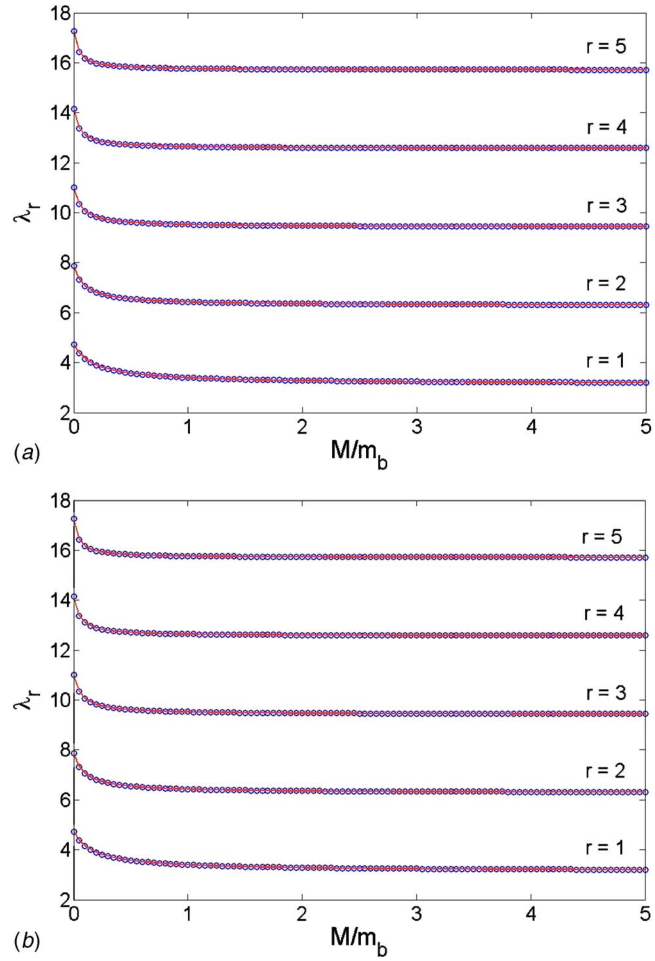

Fig. 8 Dimensionless frequency parameters versus end mass to beam mass ratio; (*a*) the first order polynomial ratio fit with the exact solution and (*b*) the second order polynomial ratio fit with the exact solution (solid line, curve fit; circle, ○ exact)

$$\omega_1 = \left(\frac{3.1416\alpha + 0.9056}{\alpha + 0.1923}\right)^2 \sqrt{\frac{EI}{m_b L^3}} \tag{32}$$

$$\omega_1 = \left(\frac{3.1416\alpha^2 + 1.821\alpha + 0.2211}{\alpha^2 + 0.4783\alpha + 0.04674}\right)^2 \sqrt{\frac{EI}{m_b L^3}} \tag{33}$$

When consistent units are used in Eqs. (32) and (33), the resulting natural frequencies will have units in rad/s, and they can be converted to hertz according to $f_1 = \omega_1 / 2\pi$. As can be seen from Figs. 6(*b*) and 7(*b*), the errors due to using Eqs. (32) and (33) instead of finding the first nonzero root of Eq. (24) are less than 0.9% and 0.05%, respectively, and these errors decrease as $\alpha = M/m_b$ increases. An important thing to note is that, to be consistent with the assumptions of Haener [4], in this work, the rotary inertias of the end masses are neglected. Therefore, one should be careful when using the above expressions for a given beam with end masses.

6 Conclusions

In this paper, it is shown that the commonly accepted fundamental natural frequency equation of a free-free thin beam with identical end masses is incorrect. Based on Euler–Bernoulli beam assumptions, the correct characteristic equation is derived and it is shown that this equation is quite different from the existing one proposed by Haener [4] in his original paper. For convenience, the results are compared in a nondimensional basis using the dimensionless frequency parameters and they are plotted against end mass to beam mass ratio. It is also shown that the expressions suggested for the higher mode natural frequencies in the mentioned work are also incorrect. Finally, the correct dimensionless frequency parameter graphs are presented and linear as well as quadratic curve fitting polynomial ratio expressions are given for the dimensionless frequency parameters of the first five modes. The resulting dimensionless frequency parameter expressions are then used to report the correct natural frequency relations for a free-free thin beam with identical end masses.

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Nomenclature

E = Young's modulus
I = cross-sectional area moment of inertia
L = length
M = each of the end masses
$M(x,t)$ = bending moment
$S(x,t)$ = shear force
f_r = natural frequency of the rth mode (Hz)
m_b = total beam mass
$w(x,t)$ = transverse displacement
x = axial position along the beam
α = end mass to beam mass ratio ($\alpha=M/m_b$)
λ_r = dimensionless frequency parameter of the rth mode
ω_r = natural frequency of the rth mode (rad/s)

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