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On the Role of Nonlinearities in Vibratory Energy Harvesting: A Critical Review and Discussion

The last two decades have witnessed several advances in microfabrication technologies and electronics, leading to the development of small, low-power devices for wireless sensing, data transmission, actuation, and medical implants. Unfortunately, the actual implementation of such devices in their respective environment has been hindered by the lack of scalable energy sources that are necessary to power and maintain them. Batteries, which remain the most commonly used power sources, have not kept pace with the demands of these devices, especially in terms of energy density. In light of this challenge, the concept of vibratory energy harvesting has flourished in recent years as a possible alternative to provide a continuous power supply. While linear vibratory energy harvesters have received the majority of the literature's attention, a significant body of the current research activity is focused on the concept of purposeful inclusion of nonlinearities for broadband transduction. When compared to their linear resonant counterparts, nonlinear energy harvesters have a wider steady-state frequency bandwidth, leading to a common belief that they can be utilized to improve performance in ambient environments. Through a review of the open literature, this paper highlights the role of nonlinearities in the transduction of energy harvesters under different types of excitations and investigates the conditions, in terms of excitation nature and potential shape, under which such nonlinearities can be beneficial for energy harvesting. [DOI: 10.1115/1.4026278]

Keywords: nonlinear, energy harvesting, monostable, bistable, low frequency, piecewise linear

1 Introduction

Humans have historically relied on harnessing ambient energy to fill their basic energy needs using windmills, sailing ships, and waterwheels. While we still rely on such techniques to fill a portion of the ever-increasing energy demand, our continuously changing technological trends necessitate adapting or even revolutionizing these old concepts. In particular, as we continue to build compact and scalable electronic devices across different fields of technology, the power requirements of such systems continue to decrease. Wireless sensors, data transmitters, controllers, and medical implants are only a few examples of technologies that have evolved in recent years to effectively function with submilliwatt power levels. For instance, wireless transponders for data transmission can now operate efficiently with less than 1 mW of power [1,2]; electronic microchips for health monitoring that consist of a sensing unit and a microcontroller have an average power consumption of approximately 50 μW [3,4].

Unfortunately, further evolution of such technologies has been hindered by the lack of a continuous scalable energy source. Batteries, which remain the most common power choice, have not kept pace with these devices' demands, especially in terms of energy density [5]. In addition, batteries must be regularly recharged or replaced, which can be very costly and cumbersome. Such challenges combined with the low-power consumption of many new critical devices have propelled several new innovations in energy-harvesting technologies. Among such technologies, micropower generators have been introduced as a new concept to transform the smallest amounts of available ambient energy into electricity [6–8]. These scalable and compact energy harvesters aim to provide a continuous power supply that permits an autonomous operation process of many electronic devices.

Within the vast field of micropower generation, vibratory energy harvesting has flourished as a major thrust area. Vibratory energy harvesters (VEHs) exploit the ability of active materials (e.g., piezoelectric, magnetostrictive, and ferroelectric) and electromechanical coupling mechanisms (e.g., electrostatic and electromagnetic) to generate an electric potential in response to mechanical stimuli and external vibrations [6–8]. Such sources of power are currently finding applications in different fields of technology, including, but not limited to, in vivo biomedical implants and health monitoring of structures and machines. For medical implants, such as pacemakers [9,10], spinal stimulators [11], and electric pain relievers [12], the availability of reliable and noninvasive power supply is of utmost importance to eliminate replacement of batteries, which has been shown to pose a significant risk of infection. It has been reported that 1.2% of the 40,000 people who annually replace batteries for pacemakers develop risky complications [13]. For structural health monitoring, vibratory energy harvesting is also envisioned to play a critical role in further evolution of these technologies. During the last two decades, more than 500 bridge failures have been reported in the United States [14], some of which, similar to the Saint Anthony Falls bridge (I-35 W) collapse in 2007, were sudden catastrophic failures claiming human lives and resulting in millions of dollars in damage. One approach to avoid such disasters centers on an early warning system using structural health-monitoring sensor networks. Traditionally, information is gathered using sensors that are hard wired to data acquisition systems. However, this conventional approach has many drawbacks, including high installation and maintenance costs. To avoid such issues, wireless health-monitoring sensor networks have been recently proposed and are currently being implemented as a replacement for the older hard-wired systems. Such networks provide similar functionalities at a

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much lower cost and, because of the absence of wires, provide higher spatial density of sensor's distribution [2]. Nonetheless, these wireless sensor networks still require a source of power, and it has been recently demonstrated that the energy harvested from vibrations caused by the flow of traffic over bridges or the swaying of a building due to wind loading provides a feasible approach to power such networks [15,16].

1.1 Vibratory Energy Harvesting: Basic Concept and Outstanding Issues. The most prolific VEH consists of a cantilever metallic beam with piezoelectric patches attached near its clamped end, as shown in Fig. 1. External environmental excitations, $a_b(t)$, applied at the base set the beam in motion, producing large strains near the clamped end, thereby straining and producing a voltage difference across the piezoelectric patches. By designing the proper circuitry, this electric potential is then used to create a current, thus converting mechanical energy from the environment to electrical energy.

In general, traditional VEHs, including the cantilever beam shown in Fig. 1, operate based on the basic principle of linear resonance. When the base excitation is harmonic with a fixed frequency, maximum energy transduction from the environment to the electric device can be achieved by tuning one of the beam's modal frequencies, usually the first, to be equal, or very close, to the excitation frequency. This resonant interaction can be used to set the beam into large-amplitude oscillations but also places critical limitations on its broadband performance characteristics. Since linear VEHs are usually designed to be very lightly damped such that the steady-state peak amplitude is maximized, their steady-state frequency bandwidth is very narrow. Therefore, manufacturing tolerances, variations in the design parameters around their nominal values, and/or variations in the nature of the excitation source can easily detune the harvester from the excitation frequency, further reducing the already small energy outputs, which limits the applicability and usefulness of VEHs that operate based on the principle of linear resonance.

The bandwidth issue becomes more pressing when one realizes that most realistic excitations seen in the environment are often not harmonic but have broadband or nonstationary (time-varying) characteristics, in which either the energy is distributed over a wide spectrum of frequencies or the dominant frequency varies with time. For instance, environmental excitations to which a bridge is subjected are generally random, resulting from wind loadings in which frequency and intensity vary depending on the atmospheric conditions and moving vehicles in which number, speed, weight, etc. vary at different times during a given day. Common sources for oscillations in microsystems have white noise characteristics due to nonequilibrium thermal fluctuations, shot, and low-frequency noise [17–19]. Thus, tuning a linear VEH to an excitation frequency becomes very challenging and usually yields inefficient transduction properties, especially outside a laboratory setting.

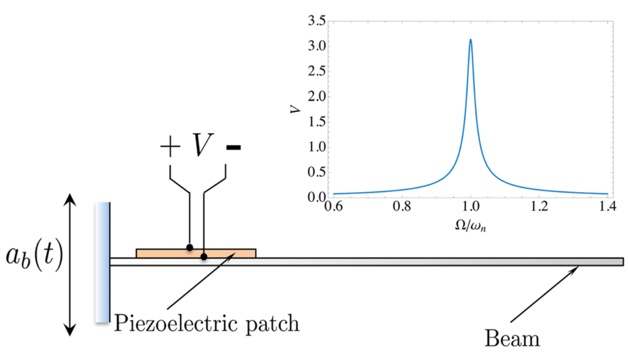


Fig. 1 Schematic of a linear cantilevered piezoelectric energy harvester and its steady-state voltage response curve. Here, $a_b(t)$ refers to the base acceleration, Ω is the excitation frequency, and ω_n is the first modal frequency of the beam.

1.2 Nonlinearity as a Solution to the Bandwidth Problem. To remedy this problem, initial solutions called for designing linear VEHs with tunable characteristics and for utilizing arrays of harvesters. Tuning mechanisms use passive/active design means to alter the fundamental frequency of the harvester to match the dominant frequency of the excitation [20–26]. Following a number of research investigations, it became evident that tunable VEHs are not efficient under random or rapidly varying frequency inputs [20]. Additionally, tuning mechanisms usually require external power or complex design means, which can reduce the efficiency of the harvester. The design of arrays of harvesters, each with a different fundamental frequency, was also proposed to allow at least one of the harvesting elements to have a matching fundamental frequency such that it resonates and harvests energy from the corresponding excitation's component [23,25,26]. This, however, reduces power density and adversely affects the scalability of the harvester.

The ability of nonlinearities to extend the coupling between the excitation and a harmonic oscillator to a wider range of frequencies has recently led many researchers to exploit them as a means to enhance the transduction of VEHs under broadband excitations. Nonlinearities can be inherently present in the dynamics of a VEH due to its geometric or material properties. Usually, they can arise from the nonlinear strain deflection relationships due to large deformations [27] or can result from a nonlinear electromechanical coupling mechanism, as in the nonlinear constitutive relationships of piezoelectricity [28]. However, these inherent nonlinearities are usually of limited value for energy harvesting because they result from the internal response characteristics of the harvester and cannot be easily controlled. More recently, the intentional introduction of nonlinearities into the design of VEHs has been a topic that received wide attention. The basic concept lies in using external design means in order to purposefully introduce and control the magnitude and nature of the nonlinearity in VEHs. The most common approach to the design of such systems introduces a nonlinear restoring force using, for example, magnetic or mechanical forces [29–32].

For the sake of demonstration, consider the same cantilever-type piezoelectric harvester shown in Fig. 1, but this time, a magnet is attached to the tip of the cantilever while a second magnet is fixed in the reference frame. These magnetic components introduce nonlinearities into the VEH, even for vibration amplitudes for which the beam itself operates in its linear regime. Under external base excitations, the tip magnet oscillates inside the potential of the other fixed magnet and the restoring force becomes a nonlinear function of the tip deflection, as shown in Fig. 2. The magnitude and nature of the nonlinearity can be altered through the design of the system. For instance, dependence of the restoring force on the tip deflection can be changed by changing the distance between the magnets or their strength. As shown in Fig. 2(*b*), the restoring force can be of the softening nature if the force decreases with the tip deflection or of the hardening type if it increases with the deflection. If the system possesses two stable equilibrium points, it is said to be bistable. As illustrated in the figure, the bistable design has a negative stiffness for small tip deflections and exhibits two additional equilibria that correspond to offset states.

Over the last couple of years, research results have indicated that, when carefully introduced, nonlinearity can be favorable for energy harvesting because it extends the bandwidth of the harvester and, hence, allows for more efficient transduction under the ambient random and nonstationary sources [29–32]. Uncertainty propagation analysis served to further reinforce these findings by showing that, under harmonic excitations, a linear device is much more sensitive to uncertainties arising from imprecise characterization of the host environment and/or from manufacturing

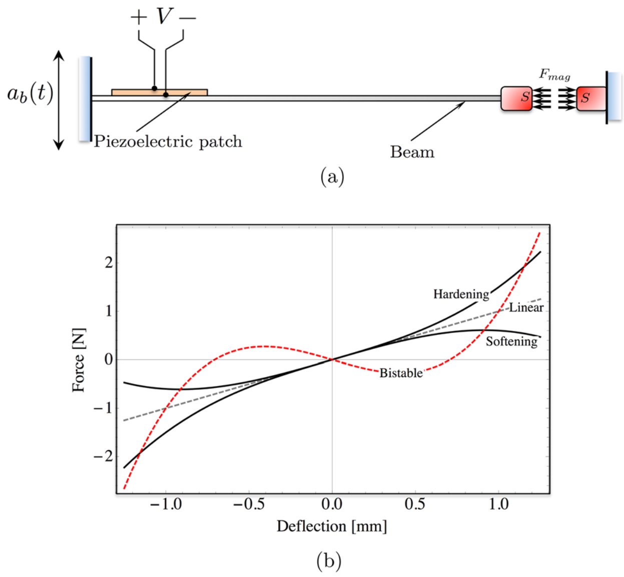


Fig. 2 (*a*) Schematic of a nonlinear cantilevered piezoelectric energy harvester. (*b*) Variation of the restoring force due to the nonlinearity.

tolerances [33]. Through a critical review of these results and detailed discussions on the influence of nonlinearities on the dynamics of mechanical oscillators, this paper reviews current research results in the literature and provides a more complete understanding of the role that nonlinearities play in the transduction of VEHs under different types of excitations. To achieve this goal, Sec. 2 introduces a basic electromechanical model that can be used to build a qualitative understanding of nonlinear VEHs. Section 3 provides a qualitative understanding of the response of mono- and bistable Duffing oscillators to harmonic excitations. This section contains the essential background necessary to understand the response of mono- and bistable VEHs to different types of excitations, which is going to be addressed in Secs. 4 and 5, respectively. Recent research efforts on incorporating a piecewise linear restoring force for enhanced transduction are presented in Sec. 6. The influence of material and coupling nonlinearities on energy harvesting, as well as the role of dissipative effects, is discussed in Sec. 7. Subsequently, Sec. 8 explores utilizing nonlinearity for energy harvesting under low-frequency excitations. Finally, a summary of the critical gaps in the current knowledge is provided in Sec. 9 with the goal of opening new avenues for research.

2 A Basic Electromechanical Model

The literature contains several lumped- and distributed-parameter models of VEHs [32,34,35]; however, these models are generally device specific and, hence, not very appropriate to build a qualitative understanding of the response behavior. To gain the insights necessary for a more general understanding, we consider a physics-based model, which can capture the qualitative behavior of energy harvesters [36]. The model consists of a lumped-parameter mechanical oscillator coupled to an electric circuit through an electromechanical coupling mechanism that is either capacitive (e.g., piezoelectric and electrostatic), Fig. 3(*a*), or inductive (e.g., electromagnetic and magnetostrictive), Fig. 3(*b*). The equations of motion can then be written in the following general form:

$$\begin{aligned} & m\ddot{\bar{x}} + c\dot{\bar{x}} + \frac{\mathrm{d}\bar{U}(\bar{x})}{\mathrm{d}\bar{x}} + \theta\bar{y} = -m\ddot{\bar{x}}_b \\ & C_p\dot{\bar{y}} + \frac{\bar{y}}{R} = \theta\dot{\bar{x}}, \text{(piezoelectric)}, \quad L\dot{\bar{y}} + R\bar{y} = \theta\dot{\bar{x}}, \text{(inductive)} \end{aligned} \tag{1}$$

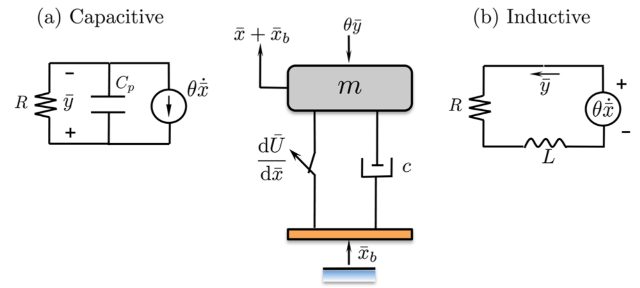


Fig. 3 A simplified representation of a generic vibratory energy harvester

where the overdot represents a derivative with respect to time, τ. The variable $\bar{x}$ represents the relative displacement of the mass, m; c is a linear viscous damping coefficient; θ is a linear electromechanical coupling coefficient; $\ddot{x}_b$ is the base acceleration; C_p is the capacitance of the piezoelectric element; L is the inductance of the harvesting coil, and $\bar{y}$ is the electric quantity representing the induced voltage in capacitive harvesters and the induced current in inductive ones. These are measured across an equivalent resistive load, R. The function $\bar{U}(\bar{x})$ represents the potential energy of the mechanical subsystem. The shape of this potential function depends on the specific nonlinearity present in the harvester but in general can be represented as

$$\bar{U}(\bar{x}) = \frac{1}{2}k_1(1-r)\bar{x}^2 + \frac{1}{4}\bar{\delta}\bar{x}^4 \tag{2}$$

which is also known as the Duffing potential, leading to cubic nonlinearities in the mechanical oscillator. Here, k_1 and k_2 are, respectively, linear and nonlinear stiffness coefficients, while r is introduced to permit variations in the linear stiffness around its nominal value. For physical realizations of most nonlinear VEHs, the introduction of this constant is necessary to reflect the fact that the linear and nonlinear stiffness coefficients cannot be changed independently. For example, when the distance between the magnets in Fig. 2 is changed, both the linear and nonlinear stiffnesses change simultaneously.

The equations of motion can be further nondimensionalized by introducing the following dimensionless quantities:

$$x = \frac{\bar{x}}{l_c}, \quad t = \tau\omega_n, \quad y = \frac{C_p}{\theta l_c}\bar{y} \text{ (piezoelectric)}, \quad y = \frac{L}{\theta l_c}\bar{y} \text{ (inductive)} \tag{3}$$

where l_c is a length scale and $\omega_n = \sqrt{k_1/m}$ is the short-circuit nominal frequency when $r = 0$. With these transformations, the nondimensional equations of motion can be expressed as

$$\begin{aligned} & \ddot{x} + 2\zeta\dot{x} + \frac{\mathrm{d}U}{\mathrm{d}x} + \kappa^2 y = -\ddot{x}_b \\ & \dot{y} + \alpha y = \dot{x} \end{aligned} \tag{4}$$

where

$$\frac{\mathrm{d}U}{\mathrm{d}x} = (1-r)x + \delta x^3 \tag{5}$$

$$\begin{array}{lll} & \text{piezoelectric} & \text{inductive} \\ \zeta = \dfrac{c}{2\sqrt{k_1 m}}, & \kappa^2 = \dfrac{\theta^2}{k_1 C_p}, & \kappa^2 = \dfrac{\theta^2}{k_1 L} \\ \delta = \dfrac{\bar{\delta} l_c^2}{k_1}, & \alpha = \dfrac{1}{R C_p \omega_n}, & \alpha = \dfrac{R}{L\omega_n} \end{array} \tag{6}$$

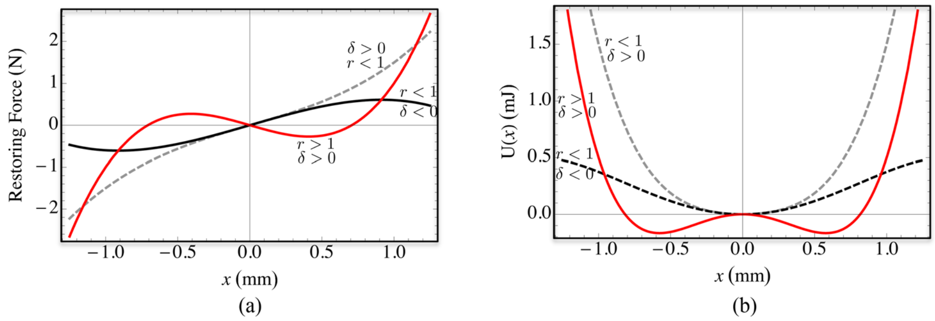


Fig. 4 Restoring force and energy potentials of different nonlinear vibratory energy harvesters

Here, ζ is the mechanical damping ratio, κ is a linear dimensionless electromechanical coupling coefficient that measures the coupling strength between the mechanical and electrical subsystems, and α is the ratio between the mechanical and electrical time constants of the harvester. This time ratio is important to characterize performance of nonlinear VEHs under random excitations. Note that κ and α have different definitions in terms of the dimensional parameters for capacitive versus inductive electromechanical coupling mechanisms. Finally, δ is the coefficient of the cubic nonlinearity. The form of the equations in Eq. (4) permits classifying energy harvesters, regardless of their coupling mechanism, into three major categories based on the shape of their potential energy function, as shown in Fig. 4.

- Linear ($\delta = 0$ and $r < 1$): In such a case, the restoring force is a linear function of the displacement, as shown in Fig. 4. Most of what are considered as linear VEHs are only linear within a certain range of operation. Large deformations and the electromechanical coupling mechanisms introduce small nonlinearities that can be usually neglected to avoid complexities in the analysis [28,37–39].
- Nonlinear monostable ($r \leq 1$): When $\delta > 0$, the restoring force increases with the displacement and is said to be of the hardening type. On the other hand, when $\delta < 0$, the restoring force decreases with the displacement and is said to be of the softening nature, as depicted in Fig. 4(*a*).
- Nonlinear bistable ($\delta > 0$ and $r > 1$): In such a scenario, the potential function of the harvester has two potential wells separated by a potential barrier, as depicted in Fig. 4(*b*). When δ is increased, the separation distance between the wells, which is defined by the location of the system's stable equilibria, $\pm\sqrt{(r-1)/\delta}$, decreases. The height of the potential barrier, $h_b = [(r-1)^2]/4\delta$, also decreases. This creates shallower potential wells, which in turn facilitates the transition of dynamic trajectories from one potential well to the other, a phenomenon commonly known as the interwell oscillations.

3 Duffing Oscillators

The basic electromechanical model developed in Eq. (4) reveals that, when the backward coupling, κ^2, is small, the influence of the circuit dynamics on the mechanical subsystem becomes negligible. In such a scenario, the dynamics of the oscillator are decoupled from the circuit dynamics with only a forward coupling effect, in other words, the mechanical oscillator influencing the harvested circuit, but not vice versa. Qualitatively, this implies that the dynamics of the harvester can be fairly well understood by studying the dynamics of the mechanical oscillator. As will be seen later in this manuscript, for a resistive load, even when κ^2 is nonnegligible, its effect on the dynamics of the oscillator can be captured by a shift in the oscillation frequency and additional linear damping [32,40]. Based on this argument, an overview of the qualitative influence of the nonlinearity on the dynamics of harmonic oscillators can go a long way towards understanding the influence of nonlinearity on vibratory energy harvesting. Readers who are not very familiar with the behavior of nonlinear systems are advised to consult an introductory book on nonlinear dynamics to better understand the concepts introduced throughout this review [41–43].

Let us consider the dynamics of a particle of mass, m, constrained to move along the path, $U(x)$, as shown in Fig. 5. For better visualization, the path can be regarded as a cart subjected to some base acceleration in the horizontal direction, $a_b(t) = A\cos(\Omega t)$. As long as the particle does not bounce throughout its motion, the restoring force is approximately proportional to the quantity $mg dU/dx$, where g is the gravitational acceleration. Thus, the shape of the path $U(x)$ determines the nature of the nonlinearity. When the path, $U(x)$, is a quartic function of x, the restoring force, dU/dx, is cubic and the motion of the particle along the path presents a Duffing oscillator. If $U(x)$ has a single well where the particle can settle at steady state in the absence of external excitation, then the Duffing oscillator is monostable. If it has two wells separated by a barrier, where the particle can settle at steady state, then it is bistable.

3.1 Frequency-Response of the Particle in the Monostable Potential.

When the path is monostable of the form, $U(x) = 1/2x^2 + 1/4\delta x^4$, and the cart is subjected to base excitations with frequency, Ω, the motion of the particle along the path can be investigated by studying variation of its steady-state amplitude $|x|$ with the frequency of excitation, also known as the frequency-response curve, as depicted in Fig. 6. When Ω is close to the local frequency of the system, ω_n, the base excitation is called a primary resonant excitation.[2] For a linear restoring force, i.e., $\delta = 0$, the frequency-response curve exhibits the typical Lorentzian-shaped behavior with large-amplitude motions occurring near the resonance frequency, i.e., $\Omega \approx \omega_n$. The associated particle responses are always unique; thus, for each frequency, the particle has a single physically realizable motion. On the other hand, when $\delta \neq 0$, the frequency-response curves bend to the left or to the right, indicating either a hardening nonlinearity, $\delta > 0$, or a softening nonlinearity, $\delta < 0$. *It is this bend in the frequency response curves that initially led researchers to exploit the nonlinearity to widen the response bandwidth of energy harvesters.* The bend in the frequency-response curves yields nonunique solutions for a certain range of frequencies and is characterized by the presence of three branches of coexisting motions, also known as attractors, namely, the large resonant branch, B_r; the nonresonant branch, B_n; and the unstable branch (dashed lines). The stable branches of solution collide with the unstable branch at two points (denoted with $s.n.$ in the figure for $\delta = -1$). These bifurcations are of the saddle-node type, where a branch of unstable saddles coalesce with a branch of stable nodes.

[2]In addition to the primary resonant excitations, nonlinear systems can exhibit secondary resonances at fraction or multiple integers of the natural frequency.

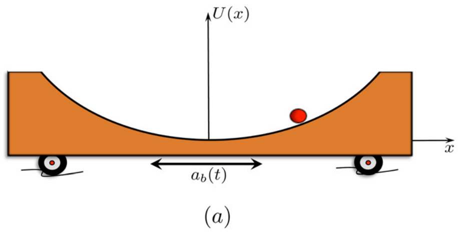


Fig. 5 The dynamics of a nonlinear energy harvester can be fairly well understood via a simple analogy with a particle moving along a cart. (*a*) Monostable potential and (*b*) bistable potential.

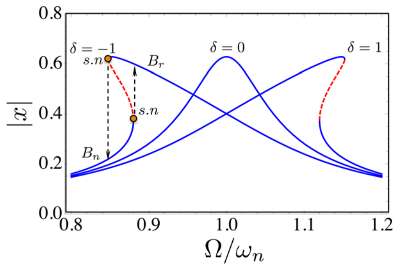


Fig. 6 Frequency response of the particle in the monostable potential

To better understand the response behavior of the particle as the frequency of excitation is varied, consider the case when δ is negative and an experiment where the excitation frequency of the cart is quasistatically increased towards resonance. The particle motion follows the nonresonant branch, B_n, of solutions up to the saddle-node bifurcation, where it jumps to the upper branch, B_r. The particle stays on the large orbit branch as the frequency is increased further. When the process is reversed, the response follows the nonresonant branch up until the higher saddle node, where it jumps down to the branch B_n and continues on that branch as the frequency is decreased further.

It is evident that the system exhibits a hysteretic behavior due to the nonlinearity. Generally, in the region of multiple solutions, the amplitude of steady-state motion depends on the direction of the frequency sweep and the initial conditions. For some set of initial conditions, the response will converge to the upper branch of solutions, while for another set, the response will approach the lower branch of solutions. The set of initial conditions leading to one solution versus the other is defined as the *basin of attraction* for the solution. For the system under study, the basin of attraction for the two solutions at different values of Ω/ω_n are shown in Fig. 7. Figure 7(*a*) depicts the basins of attraction very close to the higher saddle node, showing that the upper resonant branch has a much smaller basin of attraction. For the set of parameter values illustrated in Fig. 7(*b*), the basins of attraction for the two equilibrium states are of similar size. Finally, Fig. 7(*c*) depicts the basins near the lower saddle-node bifurcation, illustrating that the resonant branch has a larger basin of attraction in that region.

The influence of the system parameters on the steady-state frequency response is shown in Fig. 8. Increasing the forcing or decreasing the damping has a similar influence—the peak amplitude of the response increases while the bend in the frequency response remains the same. With decreases in the damping, the bandwidth of the response increases. Changing the nonlinearity, on the other hand, changes the direction and the degree by which the frequency-response curves bend but never influences the peak value, as shown earlier in Fig. 6.

3.2 Frequency-Response of the Particle in the Bistable Potential. When the path has the shape $U(x) = -1/2x^2 + 1/4\delta x^4$, $\delta > 0$, which has two minima, as shown in Fig. 5(*b*), depending on the initial conditions and the magnitude and frequency of excitation, the particle can be either confined to one potential well (*intrawell dynamics*) or can move between the two potential wells (*interwell dynamics*). For the particle to escape from one potential well, the energy supplied through the initial conditions or the external excitation should be large enough to overcome the local maximum located at the middle of the path, also known as the *potential barrier*.

In general, intrawell motions of the particle, which occur for small excitations, are less complex than their interwell counterparts. Since a single potential well is asymmetric, Fig. 9(*a*), the dynamics of the ball becomes asymmetric, as depicted in the phase portrait shown in Fig. 9(*b*). The schematic of the frequency-response curve, Fig. 10, near the local frequency of oscillation reveals that the intrawell dynamics is very similar to a monostable Duffing oscillator with softening nonlinearity. As long as the forcing amplitude A is below a critical threshold, A_1, the oscillation remains periodic with a frequency equal to that of the excitation.

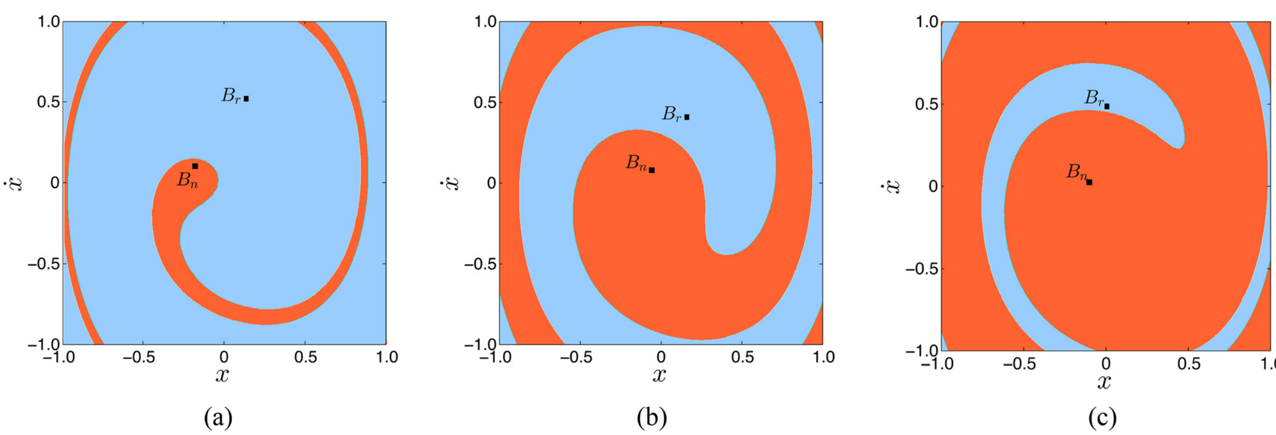


Fig. 7 Basins of attraction of the multiple solutions of a monostable system

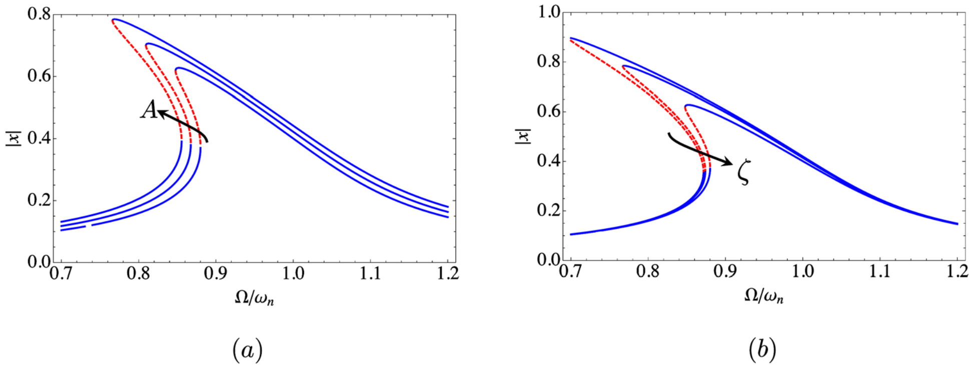


Fig. 8 Frequency response of the particle in a monostable potential for a path with softening nonlinearity. (*a*) Changing the forcing amplitude, *A*, and (*b*) changing the damping ratio, ζ.

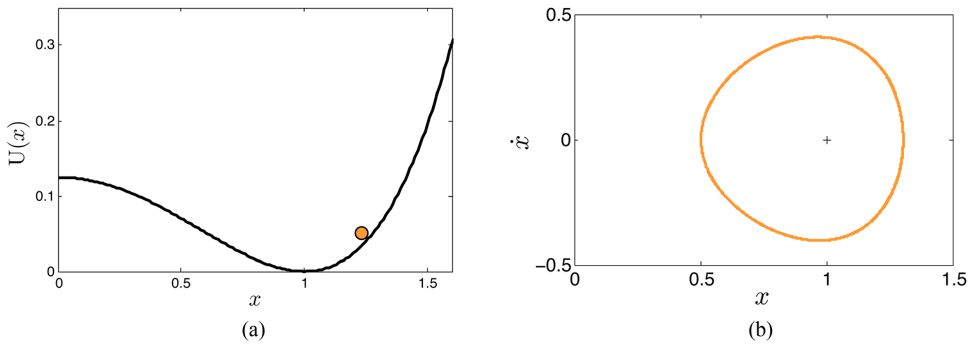


Fig. 9 (*a*) Potential energy; (*b*) intrawell oscillations

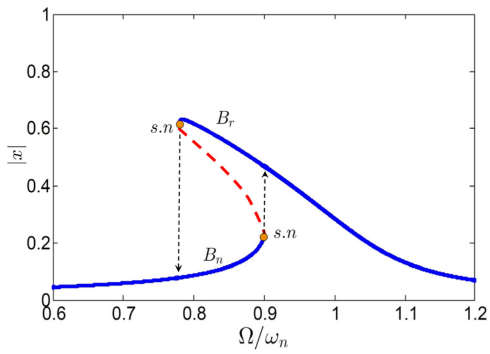


Fig. 10 Frequency response of the particle in a single potential well of the bistable potential system when $A < A_1$. Dashed lines represent unstable periodic responses.

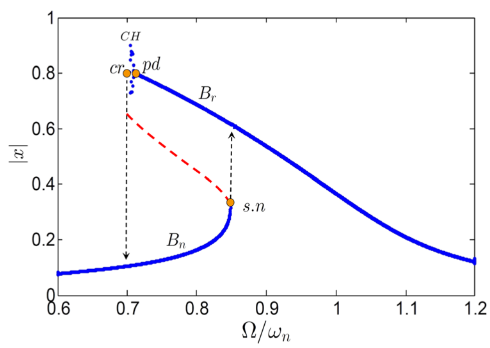


Fig. 11 Frequency response of the particle in a single potential well of the bistable potential system when $A_1 < A < A_2$. Dashed lines represent unstable periodic responses.

When the amplitude of excitation is increased beyond this threshold value, $A_1 < A < A_2$, more complex dynamic responses are observed, even when the forcing is not large enough to produce steady-state interwell motions. Specifically, it can be seen in Fig. 11 that the resonant branch of solutions undergoes a series of period-doubling bifurcations, *pd*, leading to a narrow bandwidth where chaotic responses, *CH*, occur. The chaotic attractor quickly disappears in a boundary crisis, *cr*. The saddle-node bifurcation at the coalescence of the resonant branch, B_r, and the unstable branch disappears, and the response jumps either to the nonresonant branch of solutions in the same potential well or overcomes the potential barrier and settles at the nonresonant branch of solutions associated with the opposite potential. The basins of attraction of the different solutions just above the period-doubling bifurcation and in the chaotic region are shown in Fig. 12, clearly indicating that the boundaries between the different basins become unsmooth (fractal) in nature when the response is chaotic. Therefore, any set of initial conditions in the fractal region will result in intrawell chaos and the final steady-state behavior of the ball becomes unpredictable.

When the amplitude of excitation is large enough to allow the particle to escape from the single potential well $A > A_2$, the dynamics becomes much more complex, as shown in Fig. 13. Many possible responses can exist, depending on the initial conditions and frequency of excitation. For the lower end of the frequency range considered, the small-amplitude nonresonant branch of solutions, B_n, is accompanied with the large-amplitude branch, B_L. Depending on the initial conditions, the ball can either exhibit small-amplitude intrawell motion or large-amplitude interwell oscillations. As the excitation frequency is increased towards 0.65 ω_n, the large-orbit periodic solution becomes unique, allowing the ball to undergo large-amplitude motions for any set of initial conditions. *It is this motion within the frequency response that led researchers to exploit bistability for energy harvesting*. Generally, this frequency bandwidth becomes larger as the excitation

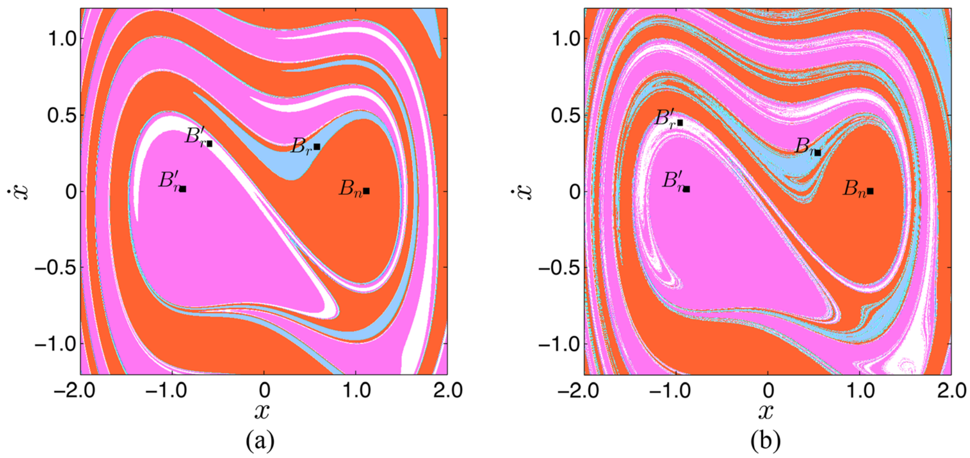


Fig. 12 Basins of attraction of the multiple solutions of a bistable system for an intermediate value of the input excitation $A_1 < A < A_2$. (*a*) Basins of attraction for a frequency ratio slightly higher than the period doubling *pd* bifurcation. (*b*) Basins of attraction for a frequency ratio within the chaotic region, *CH*. Here, B'_r and B'_n are the attractors of the opposite well.

amplitude increases. As the frequency is increased further, a chaotic attractor, *CH*, and windows of *n* period periodic solutions, namely 3*T* solutions, appear to accompany the large-orbit solution. The response of the ball in this region becomes unpredictable and very sensitive to the initial conditions.

It is evident that the response of the particle in the bistable potential is much more complex than the monostable one. Likewise, the response of VEHs with bistable potential is much more complex than their monostable counterparts. With the hope that the previous discussion is sufficient to provide the reader with a basic qualitative understanding of the response of mono- and bistable Duffing oscillators to harmonic excitations, next we present some of the major findings in the current literature with respect to the influence of nonlinearities on energy harvesting.

4 Monostable Energy Harvesters

4.1 Brief History and Examples. Early research on monostable VEHs considered the influence of inherent nonlinearities on the performance of VEHs. A correspondence by Hu et al. [37] discussed the influence of inherent geometric and material nonlinearities due to large deformations on the response behavior of piezoelectric VEHs. Their initial work provided a glimpse of how

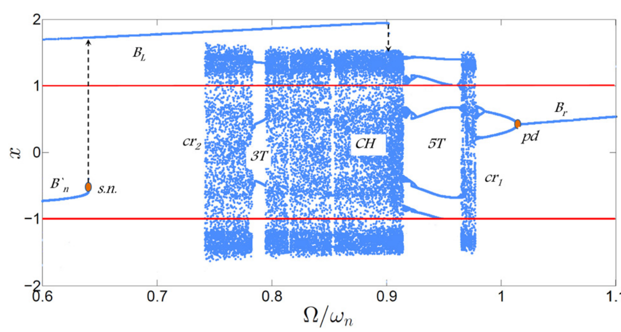


Fig. 13 Bifurcation diagram for the bistable potential system when $A > A_2$. Here, B_L represents the large-orbit branch of interwell oscillations; B_n and B'_n represent nonresonant intrawell oscillations within the two opposing wells; B_r and B'_r represent resonant intrawell oscillations within the two opposing wells; cr_i represents boundary crises; *pd* is a period-doubling bifurcation; *nT* represents periodic solutions having *n*× the period of excitation, and *CH* represents chaotic solutions.

nonlinearities can alter the behavior of VEHs and possibly provide some solutions to broaden the frequency bandwidth. Meanwhile, Beeby et al. [44] proposed a microelectromagnetic energy harvester, which, when tested experimentally, clearly exhibited a nonlinear response behavior with nonunique solutions existing for a certain frequency bandwidth. In 2007, Quinn et al. [45] also investigated the response of a generic energy harvester with essential nonlinearities to impulsive loadings, showing that the nonlinearity can be beneficial for energy harvesting.

Burrow and Clare [46] and Barton et al. [47] were among the first to intentionally introduce nonlinearities into an energy harvester and test its performance in an experimental setting. They proposed an electromagnetic VEH in the form of a tip magnet attached to a cantilever beam, as shown in Fig. 14(*a*). The magnetic potential between the magnets and the steel stator creates a nonlinear hardening restoring force. When the beam oscillates due to external excitations, the magnets move in the vicinity of a stationary steel stator and relative to a coil wound around an iron core, generating a current in the coil as per Faraday's law.

Following the work of Burrow and Clare [46] and Barton et al. [47], several other researchers have proposed other mechanisms to introduce the nonlinear compliance into monostable VEHs. In one demonstration, a magnetically levitated inductive harvester was proposed by Mann and Sims [29] and shown in Fig. 14(*b*). This harvester comprises of two outer magnets to levitate a fluctuating central magnet. The nonlinearity, which is of the hardening type, is introduced in the form of the magnetic restoring force, which also enables the system to be tuned to a specific resonant frequency. Energy is generated as a result of the relative motion between the coil and the center magnet. Stanton et al. [48] and Sebald et al. [49] also proposed a piezoelectric cantilever beam-type harvester with a tip magnet oscillating between two other stationary magnets, Fig. 14(*c*). Using this configuration, the authors showed that the effective nonlinearity of the system can be changed by altering the distance between the stationary magnets. For some distances, the harvester exhibits a softening nonlinearity, and for some others, it exhibits a hardening-type behavior, resulting in a desirable bidirectional tunability. Masana and Daqaq [32] proposed a clamped-clamped axially loaded piezoelectric beam harvester, as shown in Fig. 14(*d*). The device harvests energy as a result of the excitation-induced deformation of a piezoelectric patch attached to the surface of the beam. When the axial load is kept below the critical buckling load, the harvester exhibited a monostable Duffing-type behavior with a cubic nonlinearity in which the magnitude and nature depends on the magnitude of the axial load.

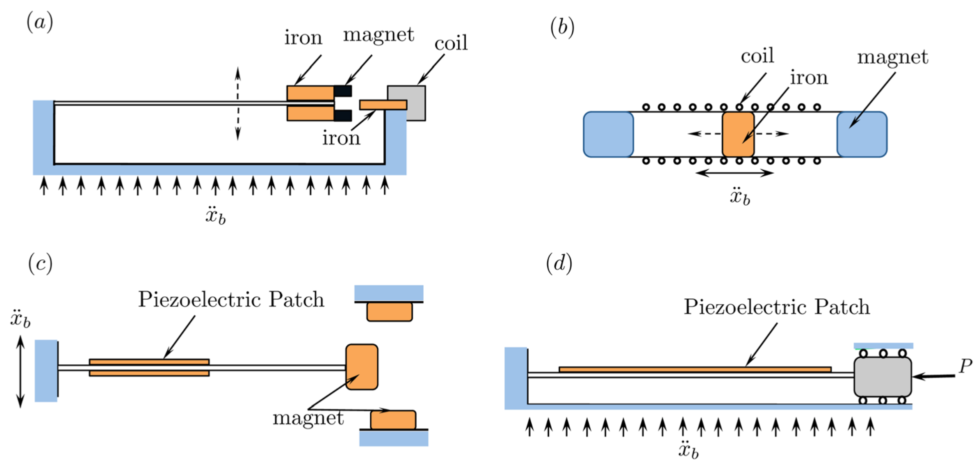


Fig. 14 Cartoon schematics of monostable energy harvester. (*a*) Inductive energy harvester proposed by Burrow and Clare [46], (*b*) inductive energy harvester proposed by Mann and Sims [29], (*c*) piezoelectric energy harvester proposed by Stanton et al. [48] and Sebald et al. [49], and (*d*) piezoelectric energy harvester proposed by Masana and Daqaq [32].

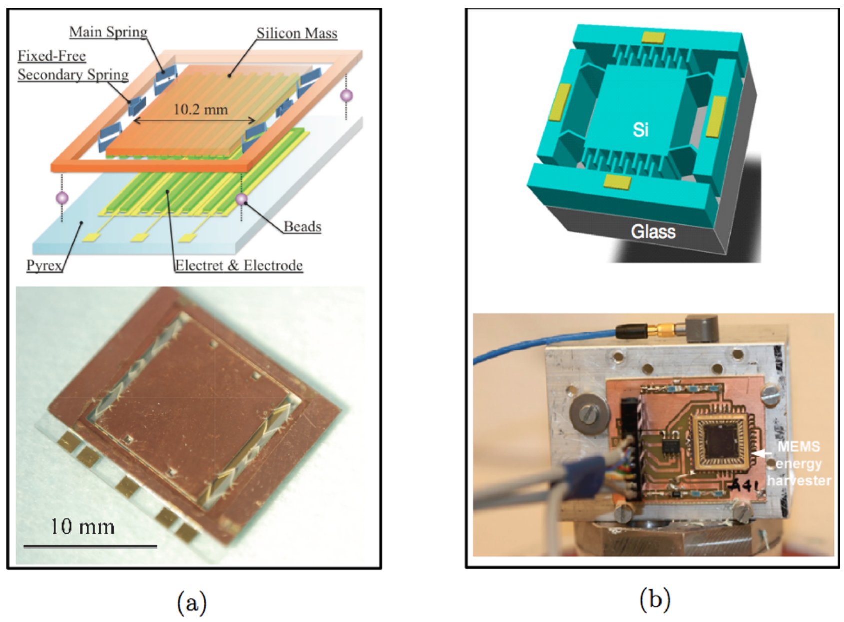


Fig. 15 Examples of capacitive monostable VEHs fabricated for MEMS applications. Part (*a*) is adapted from "Large-amplitude MEMS Electret Generator with Nonlinear Springs," by Miki et al., 2010, published in 2010 IEEE 23rd International Conference on Micro Electro Mechanical Systems (MEMS). Reproduced by permission of IEEE Publishing. All rights reserved. Part (*b*) is adapted from "Fabrication and Characterization of a Wideband MEMS Energy Harvester Utilizing Nonlinear Springs," by Nguyen et al., 2010, Journal of Micromechanics and Microengineering, 20. Reproduced by permission of IOP Publishing. All rights reserved.

In addition to the macroscale examples, several monostable energy harvesters were fabricated for microscale applications. In one demonstration, Marinkovic and Koser [50] proposed a piezoelectric device consisting of a large center clamped at its corner to four tethers, which act like clamped-clamped beams undergoing bending and stretching. The stretching effect introduces a geometric hardening nonlinearity of the monostable type, which is shown to extend the steady-state bandwidth of the harvester. Along similar lines, Tvedt et al. [51] proposed and tested an electrostatic microelectromechanical systems (MEMS) harvester consisting of a center mass clamped at its four ends using hardening suspension beams. The device also exhibited a hardening-type response similar to the behavior demonstrated in Ref. [50]. The very similar device shown in Fig. 15(*a*) was proposed by Miki et al. [52]. The proposed device exhibits a monostable Duffing-type behavior by employing nonlinear hardening springs to restore the motion of a silicon mass. Additionally, the harvester has elastic hardening stoppers that serve to constrain the motion of the device when subjected to large-amplitude excitations. These stoppers produce a piecewise nonlinear restoring force that was shown to broaden the steady-state bandwidth of the harvester. Nguyen et al. [53] fabricated and tested the electrostatic micropower generator shown in Fig. 15(*b*) with especially designed nonlinear springs that produce a softening influence. The device exhibited a softening-type response and was shown to have a wide steady-state response bandwidth.

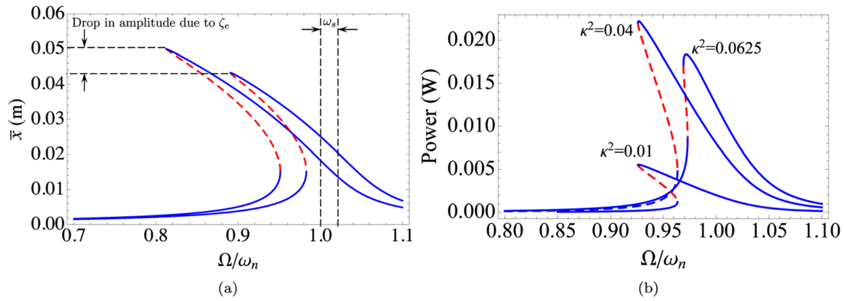


Fig. 16 (*a*) Influence of the backward coupling κ on the steady-state frequency response of the mechanical subsystem near a primary resonant excitation. (*b*) Power response of a capacitive-type monostable VEH with $\delta = -200$, $\zeta = 0.005$, $A = 0.001$, $Cp = 1\times10^{-7}$ F, $k_1 = 1000$ N/m, $\alpha = 1$, and different values of κ. Dashed lines represent unstable periodic responses.

4.2 Response to Direct Harmonic Excitations. Next, we investigate the response of nonlinear monostable VEHs to harmonic excitations. While Sec. 3.1 provided a basic picture of how nonlinearities influence nonlinear monostable oscillators, a more in-depth understanding of the influence of nonlinearity on monostable VEHs can be attained by studying the primary resonant behavior of Eq. (4) under sinusoidal base excitation of the form $\ddot{x}_b = A\sin(\Omega t)$, where Ω is very close to the natural frequency of the harvester ω_n chosen to be unity for simplicity. Using the method of multiple scales [41], an approximate analytical solution of Eq. (4) ($r = 0$) can be expressed in the form [32,54]

$$\begin{aligned} x(t) &= a(t)\cos(\Omega t - \beta(t)) \\ y(t) &= \frac{a(t)}{\sqrt{1+\alpha^2}}\cos(\Omega t - \beta(t) + \arctan\alpha) \end{aligned} \tag{7}$$

where the amplitude of the response, $a(t)$, and its phase, $\beta(t)$, are governed by the following modulation equations:

$$\begin{aligned} \frac{da}{dt} &= -(\zeta+\zeta_e)a + \frac{A}{2}\sin\psi \\ a\frac{d\beta}{dt} &= (\Omega-(1+\omega_s))a - \frac{3}{8}\delta a^3 + \frac{A}{2}\cos\psi \end{aligned} \tag{8}$$

Here, $\psi = (\Omega-1)t - \beta$. The quantity $\zeta_e = (\kappa^2\alpha)/(2(1+\alpha^2))$ represents the effective electrical damping introduced by the electric load, and $\omega_s = \kappa^2/(2(1+\alpha^2))$ represents a shift in the oscillation frequency due to the electromechanical coupling, κ. These two terms vanish when the backward coupling κ is neglected, and the dynamics of the harvester can be well understood by studying the dynamics of the equivalent monostable Duffing oscillator, as described earlier in Sec. 3.1.

The response behavior can be further simplified by investigating the steady-state amplitude and phase of the response, which can be obtained by setting the time derivatives in Eq. (8) equal to zero. These equilibrium states of the modulation equations correspond to the steady-state response of the VEH. This yields the following algebraic equations:

$$\begin{aligned} &(\zeta+\zeta_e)^2 a_0^2 + \left[(\Omega-(1+\omega_s))a_0 - \frac{3}{8}\delta a_0^3\right]^2 = \frac{A^2}{4}, \\ &\tan\beta_0 = \frac{\zeta+\zeta_e}{\Omega-(1+\omega_s)-\frac{3}{8}\delta a_0^2} \end{aligned} \tag{9}$$

The first of these represent the nonlinear frequency response equation, which can be used to study variation of the steady-state amplitude, a_0, with the excitation frequency, Ω. Upon solving this equation, the steady-state amplitude of the electric output can be written as

$$\begin{aligned} &|y| = \frac{a_0}{\sqrt{1+\alpha^2}} \Rightarrow |\bar{y}| = \frac{\theta l_c}{C_p}|y| \text{ (piezoelectric)}, \\ &|\bar{y}| = \frac{\theta l_c}{L}|y| \text{ (inductive)} \end{aligned} \tag{10}$$

and the dimensional average power dissipated in the load is

$$P = \frac{|\bar{y}|^2}{R}\text{(piezoelectric)}, \quad P = |\bar{y}|^2 R \text{ (inductive)} \tag{11}$$

The previous equations can now be used to investigate the influence of several system's parameters on the steady-state displacement and output power of the harvester. The influence of the backward coupling, κ, on the steady-state dynamics of the oscillator is depicted in Fig. 16(*a*). It is evident that increases in the backward coupling reduce the maximum amplitude of the response due to the additional energy harvested, as captured by ζ_e, and shift the frequency-response curves towards higher values of Ω due to ω_s. Thus, the influence of the backward coupling on the oscillator can be interpreted as additional linear damping together with an increase in the resonant frequency. The effect of the electromechanical coupling on the output power is shown in Fig. 16(*b*), clearly demonstrating that increases in the coupling do not necessarily increase the output power. There exists an optimal value of κ, beyond which the electrically induced damping can become too large, reducing the velocity of the oscillator and, thereby, reducing the output power. Also, since increasing the coupling increases the electric damping, the effective bandwidth of the harvester, over which the resonant branch of solutions exists, shrinks as κ increases.

The influence of the time-constant ratio, α (proportional to the load resistance R) is studied next. As shown in Fig. 17, it can be noted that the electric damping varies with α, exhibiting a maximum value near $\alpha = 1$. Near short and open circuit (large and small values of α), the electric damping is very small because little energy can be extracted from the mechanical subsystem in these conditions. When the damping is reduced, the resonant branch of solutions extends to a wider bandwidth of frequencies, as discussed earlier in Fig. 8(*b*). As such, near short and open circuit, the frequency-response curves of the oscillator extend to a wider bandwidth of frequencies, as shown in Fig. 18(*a*). The steady-state output power, which is proportional to the square of the amplitude of motion, i.e., $|x|^2$, extends exactly to the same bandwidth of frequencies as $|x|$ but gets scaled by α, with its maximum amplitude occurring near the maximum electric damping at $\alpha = 1$, as depicted in Fig. 18(*b*). Such results reveal an intricate balance between the available bandwidth of the harvester and the maximum attainable output power for a given load resistance. In particular, it is evident that, while the steady-state bandwidth is enhanced when the electric damping is reduced, the harvested

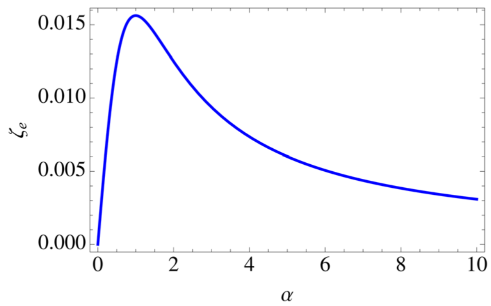


Fig. 17 Variation of the electric damping ζ_e with the time constant ratio, α

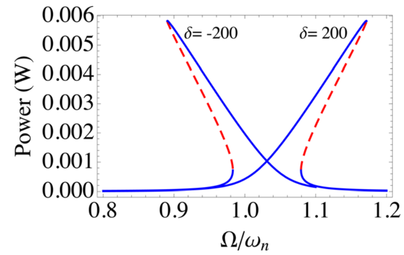


Fig. 19 Power response of a capacitive-type monostable harvester with $\kappa^2 = 0.25$, $\zeta = 0.01$, $A = 0.001$, $Cp = 1\times10^{-7}$ F, $k_1 = 1000$ N/m, and $\alpha = 1$

power is much smaller over the enhanced bandwidth. On the other hand, when the electric damping is increased, the bandwidth shrinks significantly, but the amplitude of the power is much larger at each frequency within the shrunk bandwidth.

Even though the nonlinearity increases the harvester's bandwidth when compared to a linear device, as shown in Fig. 18, the harvester is not always guaranteed to operate on the desired large-amplitude resonant branch of solutions. As described earlier in Sec. 3.1, the operating branch is determined by the basins of attraction of the coexisting solutions. Quinn et al. [55] highlighted this issue by using a probabilistic approach to estimate the steady-state response in the region where multiple solutions coexist. A weighted average value is used for the steady-state responses, with the weights calculated using the basins of attraction of the multiple coexisting solutions for physically realizable initial conditions. They illustrated that, in those regions, the probabilistic response gets closer to the smaller amplitude nonresonant branch of solutions as the frequency is shifted away from the linear resonance frequency. Several researchers have also suggested using different mechanisms to supply appropriate initial conditions that coincide with the basin of attraction of the large-orbit solution, so that the response remains on the resonant branch. However, a detailed evaluation of such mechanisms in terms of efficiency and power requirements has yet to be performed.

Sebald et al. [56] carried a detailed investigation to evaluate the performance of a monostable Duffing-type harvester. They highlighted another major issue of monostable VEHs by demonstrating that, when the nonlinearity and/or the excitation amplitude is sufficiently small, the nonlinear monostable harvester behaves, more or less, like a linear harvester, with its optimal power extracted at two frequencies, namely, the resonance frequency, $\Omega = \omega_n$, and the antiresonance frequency, $\Omega = \omega_n\sqrt{1+\kappa^2}$. They observed that the optimal time-constant ratio calculated at the optimal frequencies is dependent on the mechanical damping and electromechanical coupling and is very close to $\alpha = 1$ when the damping and coupling are small. When either the nonlinearity or the excitation amplitude are sufficiently increased, two optimal values of α appear for every excitation frequency. These optimal values correspond to maximizing the coexisting resonant and nonresonant branches of solution typically seen in the nonlinear frequency response. When the harvester is forced to operate on the resonant branch while actively changing the time constant ratio to coincide with its optimal value at each excitation frequency, the bandwidth of the output power can be extended significantly when compared to an equivalent linear harvester. However, this can be easier said than done, since the resonant branch of solutions has a very small basin of attraction as the excitation frequency shifts away from the linear resonance value. Additional difficulties are encountered, as the load resistance must be continuously optimized as the excitation frequency varies.

Finally, it is also worth noting that, for this representation of a VEH, the nonlinearity by itself can only be used to bend the response curves but has no influence on the amplitude of the power, as shown in Fig. 19.

4.3 Response to Random Excitations. While most environmental excitations under which VEHs are designed to operate have random or time-dependent characteristics, their design and optimization is currently, for the most part, based on steady-state analyses under harmonic excitations, as discussed in Sec. 4.2. While this constitutes an important first step, it does not provide the tools or insights necessary for improving their performance in an actual environment. Recently, various researchers realized the importance of understanding the influence of the nonlinearity on the transduction characteristics of VEHs under white and colored random excitations. Among these studies, the response of monostable Duffing-type harvesters to random excitations has been analyzed by few researchers [30,57,58].

4.3.1 White Noise. The influence of the nonlinearity on the performance of monostable Duffing VEHs under white Gaussian excitations has been investigated by various researchers [30,53,58–60]. It was determined that the time-constant ratio, α, plays an important role in characterizing the influence of the nonlinearity. Gammaitoni et al. [30] numerically and experimentally studied the response of a monostable piezoelectric energy harvester to random excitations. They showed that, when the time

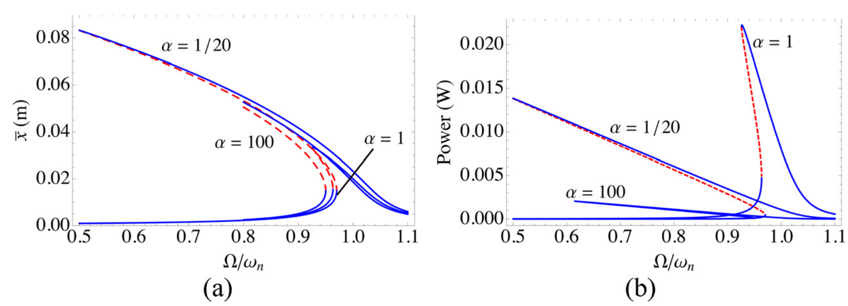


Fig. 18 Frequency and power response of a capacitive-type monostable harvester with $\delta = 200$, $\zeta = 0.005$, $A = 0.001$, $Cp = 1\times10^{-7}$ F, $k_1 = 1000$ N/m, $\theta = 0.002$ N/Volt

constant of the harvesting circuit $1/(RC_p)$ is very small, i.e., α is very large, the root mean square (rms) output voltage always decreases with the nonlinearity for a fixed linear stiffness. In 2010, Daqaq [58] considered an inductive monostable energy harvester and formulated the Fokker-Plank-Kolmagorov (FPK) equation governing the evolution of the probability density function of the harvester's response under white Gaussian noise. He showed that, when the inductance of the coil can be neglected (equivalent to having a very large time constant ratio, α), the PDF of the response can be separated into a function of the displacement and a function of the velocity. He proved that, under such conditions, the output power of the harvester is not a function of the nonlinearity. Therefore, linear and nonlinear monostable harvesters produce exactly similar power levels under white noise excitations provided that the time-constant ratio between the mechanical oscillator and the harvesting circuit is very large. Sebald et al. [56] confirmed these results experimentally by showing that the output power levels of a linear and a nonlinear monostable Duffing-type harvester are very close when both are excited with equivalent broadband noise.

In an extension to his earlier work, Daqaq [36] also showed that, even when the time-constant ratio is not very large for both capacitive and inductive harvesters, the output voltage decreases with the nonlinearity as long as it is of the hardening nature. He concluded that, for two energy harvesters with equal linear stiffnesses, the one with zero nonlinear stiffness component always outperforms the one exhibiting a hardening nonlinear behavior. Recent research results by Green et al. [61] corroborated these findings but also showed that, although both the linear and nonlinear harvester's produce exactly similar power levels under white noise, the harvester with the nonlinear restoring force has a smaller rms displacement when compared to the linear one, making it better suited for applications with constrained space. However, such conclusions should be treated carefully, since the reduction of the rms value of the displacement does not necessarily prevent the instantaneous displacement from being large. In a recent study, Halvorsen [60] also demonstrated that the rms voltage of the harvester is not a function of the nonlinearity when the time constant of the harvesting circuit is very small. He showed that, for intermediate values of the time constant, the rms voltage of a monostable harvester with a hardening nonlinearity can never be larger than that of a linear harvester with equal linear stiffness. On the other hand, a monostable harvester with a softening nonlinearity can produce more power than a linear harvester with equal linear stiffness [53]. The results of Halvorsen [60] have been further confirmed by He and Daqaq [62], who showed that asymmetries in the restoring force due to softening quadratic nonlinearities can provide performance enhancement under white noise over an equivalent linear harvester.

The optimization of the electric load for monostable Duffing harvester's under white noise was discussed by several researchers [61,62]. Green et al. [61] considered an electromagnetic energy harvester with cubic-hardening nonlinearities and used statistical linearization to show that, when neglecting the inductance of the coil (α approaches infinity), the optimal load is not a function of the nonlinearity and is equal to that corresponding to the optimization of the linear problem. He and Daqaq [62] generalized the optimization problem to any generic nonlinear monostable VEH with quadratic and cubic nonlinearities and any time-constant ratio. They illustrated that, when the time-constant ratio is not very large, the optimal load is in fact a function of the nonlinearity.

4.3.2 Colored Noise. Barton et al. [47] experimentally analyzed the response of the nonlinear electromagnetic energy harvester shown in Fig. 14(*a*) to narrow-band random excitations. The narrow-band excitation was created by passing white Gaussian noise through a bandpass filter with a predefined bandwidth and center frequency. They studied variation of the rms tip velocity of the harvester with the center frequency of the random excitation for different bandwidths. They observed that, in the region where multiple solutions coexist in the nonlinear steady-state response curves, the harvester cannot maintain motions on the higher orbit resonant branch under the band-limited forcing. The response jumps between the two branches of solution, causing the output voltage to drop significantly when compared to the steady-state response attained under harmonic excitations. The authors concluded that the bend in the steady-state frequency response curves does not enhance the performance of the harvester under band-limited random excitation. Sebald et al. [49,56] arrived at a similar conclusion for a hardening-type monostable harvester. Nguyen and Halvorsen [63] demonstrated that a harvester with a softening nonlinearity outperforms an equivalent linear harvester if the center frequency of the excitation is tuned below the linear oscillation frequency. The enhanced performance can be attained regardless of the harvester's bandwidth.

Daqaq [58] investigated the response of monostable Duffing-type inductive harvesters to band-limited random excitations. He used the Van Kampen expansion to obtain approximate analytical solutions for the FPK equation governing the response statistics and found that, when the noise is centered at the natural frequency of the harvester, the power always decreases with the nonlinearity and that reduction in the power is most pronounced for excitations with smaller bandwidths. He also showed that, enhancing the performance of Duffing-type harvesters under band-limited noise requires tuning the center frequency of the excitation to be above the natural frequency of the harvester for hardening-type nonlinearities and vice versa for softening ones. Lee et al. [64] verified this result experimentally and showed that enhanced performance is mostly pronounced for excitations with smaller bandwidths and when the nonlinear device is subjected to constant perturbations that push the harvester to operate near the resonant branch of solutions.

4.4 Response to Other Types of Excitations. The response of monostable VEHs to other common ambient excitations has also been considered. In one demonstration, Daqaq et al. [65] investigated the performance of a piezoelectric cantilever-beam-type VEH to parametric excitations. As shown in Fig. 20(*a*), when a cantilever beam is excited parallel to its length, a parametric instability can be activated when the magnitude of the forcing exceeds a certain threshold and the excitation frequency is close to twice the fundamental frequency of the harvester [41]. Using a single-mode reduced-order model, Daqaq et al. [65] showed that the region of parametric instability wherein energy can be harvested shrinks as the coupling coefficient, κ, of the harvester increases and that there exists an optimal coupling coefficient beyond which the peak power decreases. They also demonstrated that there is a critical excitation level below which no energy can be harvested. The amplitude of this critical excitation increases with the coupling coefficient and exhibits a maximum value at a given load resistance. Abdelkefi et al. [66] extended the work of Daqaq et al. [65] by considering the multimodal behavior of the same harvester and showed that a single-mode approximation underestimates the actual output power of the device. In another demonstration, Ma et al. [67] considered a parametrically excited pendulum-inductive generator, as shown in Fig. 20(*b*), and showed that such device is ideal for harvesting energy from low-frequency excitations.

Due to inherent system nonlinearities, many vibratory excitation sources possess a frequency spectrum that contains energy components at multiple integers of the fundamental frequency of the source. To capture energy from these frequency components, Daqaq and Bode [68] exploited the parametric amplification phenomenon to channel energy from the excitation's superharmonics, namely the one at twice the fundamental frequency, to a purely resistive load. To achieve this goal, they considered a piezoelectric cantilevered-type bimorph harvester and showed that, by tilting the axis of the beam through a proper angle with respect to the

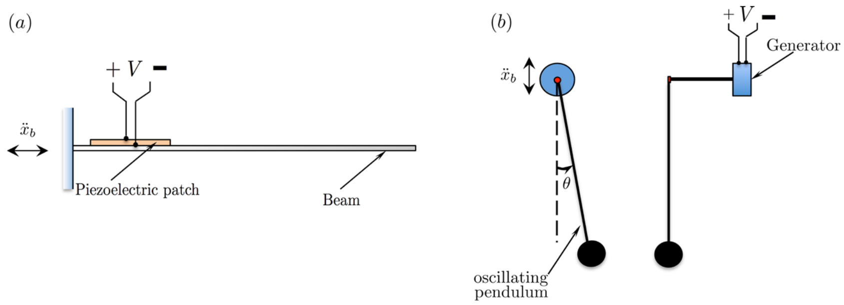


Fig. 20 A schematic of parametrically exited energy harvesters considered by (*a*) Daqaq et al. [65] and (*b*) Ma et al. [67]

direction of excitation, it is possible to utilize a parametric pump to enhance the output power at the fundamental frequency. Percentage improvement in the output power was shown to depend on the excitation's parameters and the mechanical damping ratio. When the mechanical damping ratio is small, significant enhancement in the output power is attainable, even when the magnitude of the superharmonic is small when compared to the fundamental frequency.

Quinn et al. [45,54] investigated the response of a monostable VEH to impulsive loads and showed that, although a linear device has better performance at its fundamental frequency, the nonlinear device outperforms the corresponding linear system in terms of both magnitude of power harvested and the frequency interval over which significant power can be drawn when the excitation is of the impulsive type.

4.5 Apparent Issues. Based on the previous discussions, it is apparent that the intentional inclusion of nonlinearities in monostable energy harvesters makes the device more tolerant to variations in the excitation frequency around its nominal value when compared to a linear device. This idea has been highlighted by Quinn et al. [55], who showed that, for certain potential shapes obtained by carefully optimizing the linear and nonlinear stiffness components, it is possible to design a VEH with enhanced bandwidth that can account for small drifts in the excitation frequency. Mann et al. [33] have also demonstrated using uncertainty propagation analysis that, under harmonic excitations, a linear device is much more sensitive to uncertainties arising from imprecise characterization of the host environment and/or from manufacturing tolerances. For certain types of excitations, especially those that have an impulsive nature or slowly time-varying frequencies, there seems to be potential benefits in utilizing a monostable Duffing-type VEH to improve performance. However, beyond accounting for small drifts in the excitation frequency, performance improvements via the intentional inclusion of nonlinearities in a monostable Duffing harvester is questionable. Following are the major issues:

- Nonuniqueness of solutions: As discussed earlier, the resonant large-amplitude branch favorable for energy harvesting is always accompanied by a smaller-amplitude branch. The ability of the harvester to operate on the upper branch of solutions is determined by the initial conditions and its basin of attraction. Due to its larger basin of attraction away from resonance, probabilistic studies seem to suggest that, on average, the output voltage will be closer to the lower branch of solutions as the frequency shifts away from the linear resonance value, which diminishes the importance of the resonant branch of solutions. Additionally, techniques proposed to guarantee operation on the resonant branch seem to consume additional power and still need further evaluation.
- Excitation levels: When the excitation level is small, the influence of the nonlinearity diminishes and a monostable Duffing harvester loses its broadband properties, behaving similar to a linear resonant device. Thus, no performance gains are expected by incorporating nonlinearities when the level of excitation is small.
- Balance between electric damping and bandwidth: It has been made clear in Sec. 4.2 that optimizing the electric load to maximize the electric damping improves the output power but reduces the bandwidth, making the harvester less tolerant to frequency variations. Even when the load resistance is optimized as a function of the frequency, which yields improved bandwidth over a linear harvester, this still requires changing the load in real time as a function of the excitation frequency, which is very hard to implement in a real setting.
- Nonlinearity and maximum attainable power: It has been implicitly suggested by some researchers that nonlinearity can be used to increase the maximum attainable power of the harvester under steady-state harmonic excitations. It is essential to make it clear that, for the systems commonly considered, the nonlinearity by itself can only bend the frequency-response curves but not increase their amplitude (Fig. 19). The increased power observed by some researchers in experimental settings could be due to variations in other design parameters, e.g., linear stiffness and electromechanical coupling, as the nonlinearity is changed.
- Response to random excitations: The bandwidth tolerance properties of monostable VEHs under steady-state harmonic excitations is by no means an indication of enhanced performance under other common types of nonstationary and random environmental excitations. Research results suggest that hardening-type nonlinearities cannot be used to improve performance under random excitations that can be approximated by a white noise process. However, softening-type nonlinearities that result in an asymmetric potential function may help enhance performance under such excitations.

5 Bistable Energy Harvesters

5.1 Brief History and Examples. The first investigation of bistability in the energy-harvesting literature appears to be a theoretical paper by McInnes et al. [69] who reported the benefits of the stochastic resonance (SR) phenomenon for performance enhancement in vibratory energy harvesting. The concept of SR was introduced in the early 1980s by Benzi et al. [70] to explain the dramatic oscillations in the Earth's long-term climate change and thereafter found many applications in weak signal amplification by noise addition in the presence of a periodic excitation component [71–74]. Building on this background, in 2008, McInnes et al. [69] proposed the use of SR to boost the harvested power by adding periodic forcing to noise excitations. One year later, the first two bistable energy harvesters and describing qualitative models were published independently by Cottone et al. [75] and Erturk et al. [76]. Both groups created the bistable restoring force by using different magnet arrangements to induce a magnetoelastic buckling in a piezoelectrically laminated beam. For

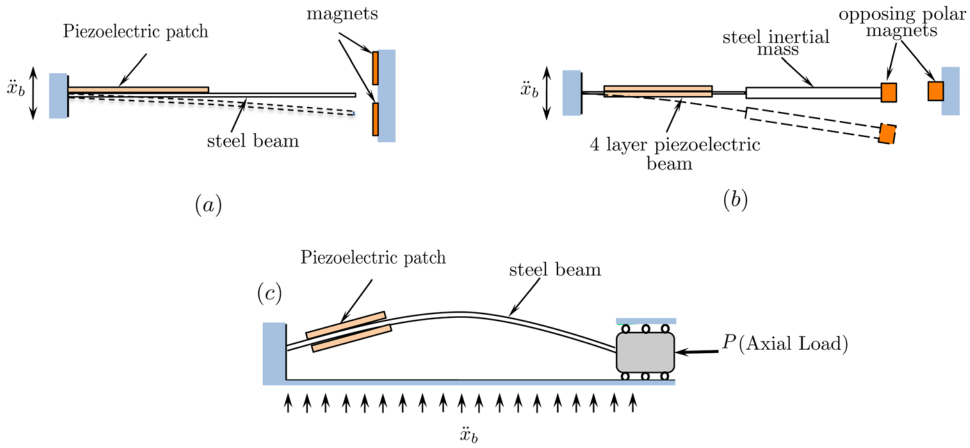


Fig. 21 Schematics of the bistable piezoelectric energy harvester configurations suggested by (*a*) Erturk et al. [76], (*b*) Cottone et al. [75], and (*c*) Masana and Daqaq [86]

instance, as depicted in Fig. 21(*a*), the device proposed by Erturk et al. [76] simply added piezoelectric laminates to the bistable magnetoelastic cantilever arrangement suggested by Moon and Holmes [77] in their early work on chaotic vibrations in structural mechanics. Similarly, as shown in Fig. 21(*a*), Cottone et al. [75] and Gammaitoni et al. [30] used magnets to create a bistable restoring force in the same vein as their previous work on bistable systems [72,73].

Following the early efforts on the use of magnetoelastic potentials for creating bistability [75,76], researchers have proposed other methods, including purely elastic buckling due to axial loads in beams [78–80] as well as laminate asymmetry in the case of composite plates [81]. For instance, Arietta et al. [81] demonstrated that certain laminations of carbon-fiber-epoxy plates can result in buckling due to the different thermal expansion coefficients at room temperature [82]. Therefore, in such cases, it is not even required to impose an external force or a magnetic field to create a bistable restoring force. Mann and Owens [83] also used a smart arrangement of magnets to illustrate that bistability can be created without even utilizing the buckling of elastic structures.

In addition to these efforts at the macro/meso scale, researchers have also started using bistable architectures at the MEMS scale. Ando et al. [84] were able to successfully implement bistability for broadband energy harvesting by magnetoelastic buckling using a MEMS version of the configuration proposed by Cottone et al. [75] in Fig. 21(*a*) for piezoelectric energy harvesting. Most recently, Nguyen et al. [85] fabricated and tested MEMS electrostatic energy harvesters with curved springs to demonstrate bistability and substantially enhanced frequency bandwidth. They reported broadband power output for different bias voltage levels across the capacitor fingers.

Since 2009, bistable vibratory energy harvesting formed the core of many theoretical and experimental investigations by several other researchers [48,59,78,79,81,83–92]. Both deterministic [48,78,79,81,83–86,90,91] and stochastic [58,85,87–89,92] excitations have been studied using different transduction mechanisms that include piezoelectric [48,78,79,81,86–92], electromagnetic [58,83,91], and electrostatic for MEMS implementations [84,85]. For the most part, these efforts suggested that bistable VEHs can potentially outperform their linear counterparts. However, the performance metrics used to quantify these improvements were not clearly specified. As a result, conclusions on performance enhancement lacked solid evidence and were often generalized. In some instances, broadband steady-state bandwidth under harmonic excitations was used to erroneously predict improved performance under random and nonstationary excitations. For instance, a bistable harvester that performs well under certain types or levels of harmonic excitations might not perform as well under random or nonstationary input. One reason for such confusion stems from the complex response behavior of bistable energy harvesters as compared to their linear or even nonlinear monostable counterparts. Additionally, lack of analytical tools that permit characterizing their performance under harmonic and random excitations has limited most of the literature to numerical and experimental studies. In the following sections, we hope to draw a clearer picture of how bistability influences performance of VEHs under harmonic and random excitations.

5.2 Response to Harmonic Excitations. Erturk et al. [76,93,94] were among the first to investigate the response of bistable VEHs to harmonic excitations. They pointed out that the main advantage of a bistable VEH is in the presence of the large-orbit solution resulting from the interwell oscillations that can be excited for some excitation levels. They clearly illustrated that the large-orbit response associated with interwell oscillations can yield substantially larger power output over a wider bandwidth of frequencies when compared to an equivalent linear VEH, Fig. 22(*c*). However, as shown in Figs. 22(*a*) and 22(*b*), it was also observed numerically and experimentally that this large-orbit branch is not always unique and can be accompanied by a chaotic attractor and small branches of undesired intrawell oscillations.

Stanton et al. [48] arrived at similar conclusions while rigorously investigating amplitude and frequency bifurcations of an alternative bistable piezoelectric VEH similar to the one shown in Fig. 14(*c*). Unlike the qualitative numerical model of Erturk et al. [76] and Erturk and Inman [93], Stanton et al. [48] represented the dynamics of the coupled system quantitatively using a nonlinear analytical model. Again, coexisting large- and small-orbit periodic and chaotic responses were reported for different levels of the input harmonic excitation. Similar results were also reported for different systems and configurations [83,90,91], indicating the ability of a bistable VEH to produce high power levels for a range of frequencies and input excitations, but also clearly highlighting the complexity and nonuniqueness of the resulting voltage responses.

While these initial results highlighted the ability of bistable VEHs to produce large-amplitude responses under harmonic excitations for some excitation levels and frequency ranges, they also pointed out many issues resulting from the complexity and nonuniqueness of the resulting solutions. It also became apparent that the performance of a bistable VEH is very much dependent on the level of the excitation and the shape of the potential energy function of the harvester. As discussed earlier in Sec. 3.2, the response of the harvester can be confined to one potential well, if the excitation level is too small to permit interwell oscillations. In such a scenario, the harvester behaves most or less similarly to a monostable VEH with softening nonlinearity with no potential

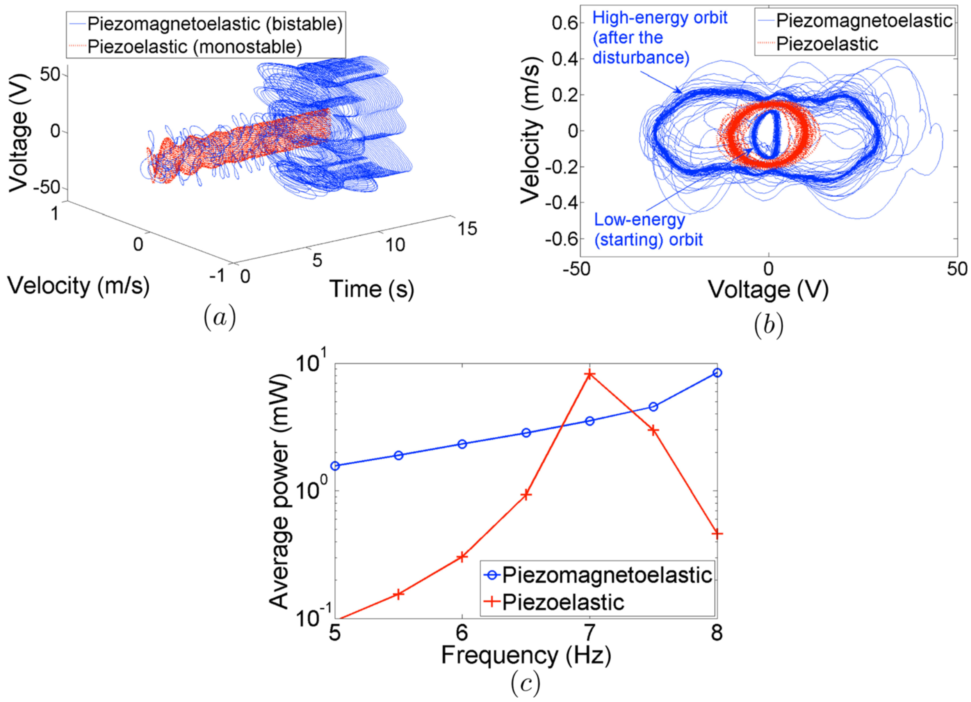


Fig. 22 Experimental voltage versus velocity trajectories of bistable (piezomagnetoelastic) and monostable (piezoelastic) energy harvesters showing the advantage of the high-energy (interwell) orbits in the bistable harvester, which may coexist with (*a*) chaotic and (*b*) low-energy (intrawell) periodic attractors; (*c*) power frequency response curves of bistable and monostable energy harvesters for the same excitation level [76,93]

advantages over a nonlinear monostable Duffing VEH. Furthermore, as pointed out in Sec. 3.2, for excitation levels that are large enough to allow transient interwell oscillations but not sufficiently large to permit steady-state periodic interwell oscillations, the response remains for most of the time confined to one potential well and might exhibit intrawell chaos. For such excitations, there is no clear indication that the bistable VEH provides measurable advantages. Only when the excitation is sufficiently large to maintain periodic interwell oscillations, a bistable VEH can maintain motion on the desired large-orbit branch of solutions, which could potentially provide advantages over a linear device.

During their work on the bistable buckled beam VEH shown in Fig. 21(*c*), Masana and Daqaq [78] further highlighted the influence of the potential shape in conjunction with the excitation level on the performance of bistable VEHs. Their VEH was first designed such that the buckling load is large enough to produce deep potential wells. The VEH was subjected to harmonic excitations of different levels and frequencies, as shown in Fig. 23(*a*). For the smallest level of excitation, the dynamics remained confined to one potential well, with a very narrow band of intrawell chaos appearing similar to what is observed for a bistable system in Fig. 11. Obviously, this bistable design is not suitable for harvesting energy from low-level harmonic excitations. As the excitation level is increased to a medium level, e.g., $10\,\mathrm{m/sec^2}$, the cross well chaotic band extended to a wider range of frequencies, but the complexity of the output voltage response made it generally inappropriate for energy generation. Only when the excitation level was further increased to higher values, the desired large-orbit branch of periodic solutions became visible and extended over a wide band of frequencies, producing the desired large-amplitude periodic voltages at every frequency of excitation. For this high level of excitation, the bistability can indeed provide a more bandwidth-tolerant VEH with enhanced performance under harmonic excitations.

The same VEH was then redesigned such that the buckling load produces shallow potential wells. Again, the VEH was subjected to the same harmonic excitations as in the previous case. It was observed that the large-orbit branch of periodic solutions can now be activated, even when the excitation level is very small, making it more suitable for lower levels of excitation, as shown in Fig. 23(*b*). As the excitation level was increased, the large-amplitude branch of solutions extended to a large band of frequencies. While it is tempting to generalize, these results do not necessarily imply that further reduction in the depth of the potential wells is always desirable. Further reduction can make the system lose its bistable properties, making it less effective, especially for higher excitation levels. A similar bistable energy harvester configuration made of a buckled thin beam with clamped end conditions was also investigated by Sneller et al. [79]. They added a lumped central mass to the buckled beam and showed that the inclusion of the mass reduces the excitation threshold necessary to activate the interwell motions and broadens the frequency range over which these motions exist.

Even when the excitation is large enough to excite the interwell oscillations, a common challenge lies in the nonuniqueness of these stationary solutions [48,76,83,93]. For an excitation level that might yield high-energy responses, it is still possible to have coexisting lower energy branches of solutions with the final steady-state behavior determined by the solutions' basin of attraction. For instance, for the buckled beam harvester proposed by Masana and Daqaq [78], the large-orbit branch of solutions, B_L, shown in Fig. 24, is accompanied by a lower periodic orbit, B_n, for the lower range of the frequency spectrum, followed by a region where the solution is unique and by either a chaotic attractor, *CH*, or other windows of lower-amplitude *n* periodic voltage responses as the frequency is increased further. To overcome this challenge, it was suggested by Erturk et al. [76] that the piezoelectric layers can be used for impulse actuation to create a disturbance, which changes the response state (from low energy to high energy) potentially by discharging a capacitor (which would be compensated for by harvesting energy for long enough time). As of today, however, no physical system has been built in the

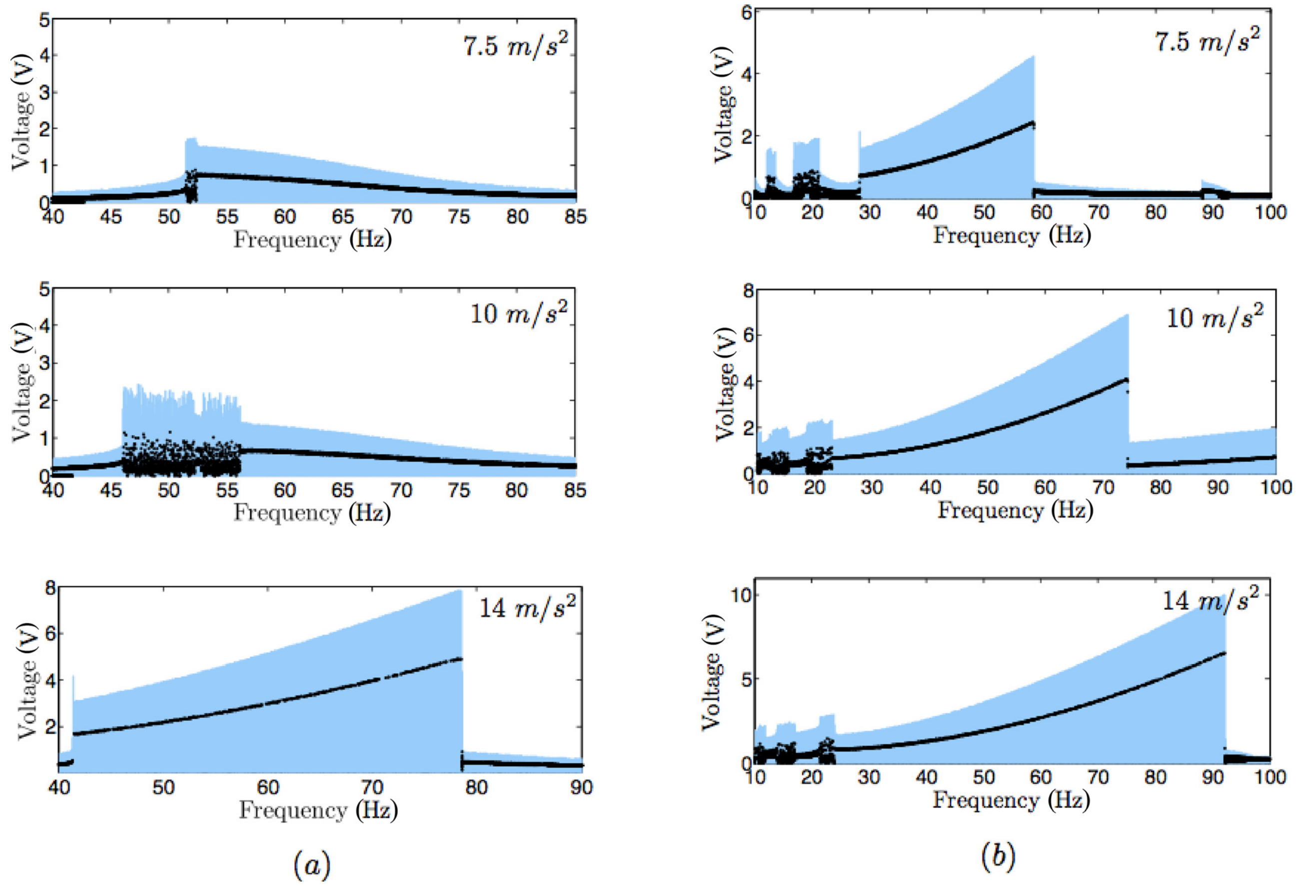


Fig. 23 Numerical frequency-response curves of the bistable energy harvester proposed by Masana and Daqaq [78]. (*a*) A bistable potential with deep wells. (*b*) A bistable potential with shallow wells. Dots represent a bifurcation map and solid regions represent the amplitude of the steady-state time history.

literature to ensure operation on the high-energy orbit of coexisting solutions in a nonlinear VEH. For instance, the chaotic response in Fig. 22(*a*) and the intrawell response in Fig. 22(*b*) were switched to large-amplitude periodic response by manually applying the disturbance after 10 seconds.

Another way to overcome this nonuniqueness problem is to extend the bandwidth of frequencies where the unique large-orbit periodic solutions exist. This, however, cannot be done without

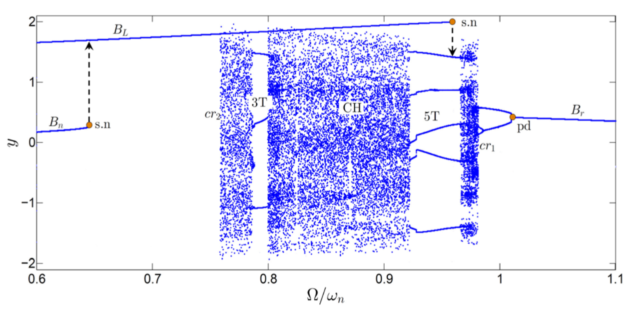


Fig. 24 Voltage frequency response of a bistable axially loaded VEH near its primary resonance. B_L represents large-orbit branch interwell oscillations, B_n represents nonresonant intrawell oscillations, B_r represents resonant interwell oscillations, cr_i represents a boundary crisis, *pd* is a period-doubling bifurcation, *nT* represents solutions having $n\times$ the period of excitation, and *CH* represents chaotic solutions. Results are obtained for Eq. (4) with $\zeta = 0.05$, $r = 1.5$, $\delta = 0.5$, $\kappa^2 = 0.01$, $\alpha = 0.1$, and a base excitation $A = 0.175$.

understanding the influence of the system parameters on the response behavior. Depending only on numerical simulations might not be sufficient to resolve this dependence, as numerical results rarely provide a clear insight into how system parameters influence the behavior of complex systems. This has motivated researchers to exploit some analytical or semianalytical tools to understand this complex problem [95–97]. Among those researchers, Stanton et al. [95] and Harne et al. [97] used the method of harmonic balance to construct analytical solutions that approximate the amplitude and stability of the periodic voltage responses resulting from a bistable harvester. In Fig. 25, we show variation of the steady-state output voltage with the frequency of excitation as obtained using the method of harmonic balance for three different excitation levels. To obtain these solutions, we retained only the fundamental harmonic and a constant shift in the assumed solution. Thus, only stable and unstable solutions of period one are depicted in the figure. The figure illustrates that there are three different critical frequencies on the output voltage response. The first represents a period-doubling bifurcation, *pd*, occurring on the resonant branch of intrawell motions. This point triggers a cascade of period-doubling bifurcations that leads to a chaotic attractor, which quickly disappears in a boundary crisis, causing the harvester to operate on the desired large-orbit interwell branch, B_L. Thus, the period-doubling bifurcation provides a rough overestimate for the highest frequency value at which the harvester might be able to operate on the desired large-orbit branch of solutions. The second point represents the frequency value at which the large-orbit branch of solutions disappears in a saddle-node bifurcation, *sn*2. When this frequency is larger than *sn*1, it can be used to estimate the lowest value for which the large-orbit branch of interwell oscillations is unique.

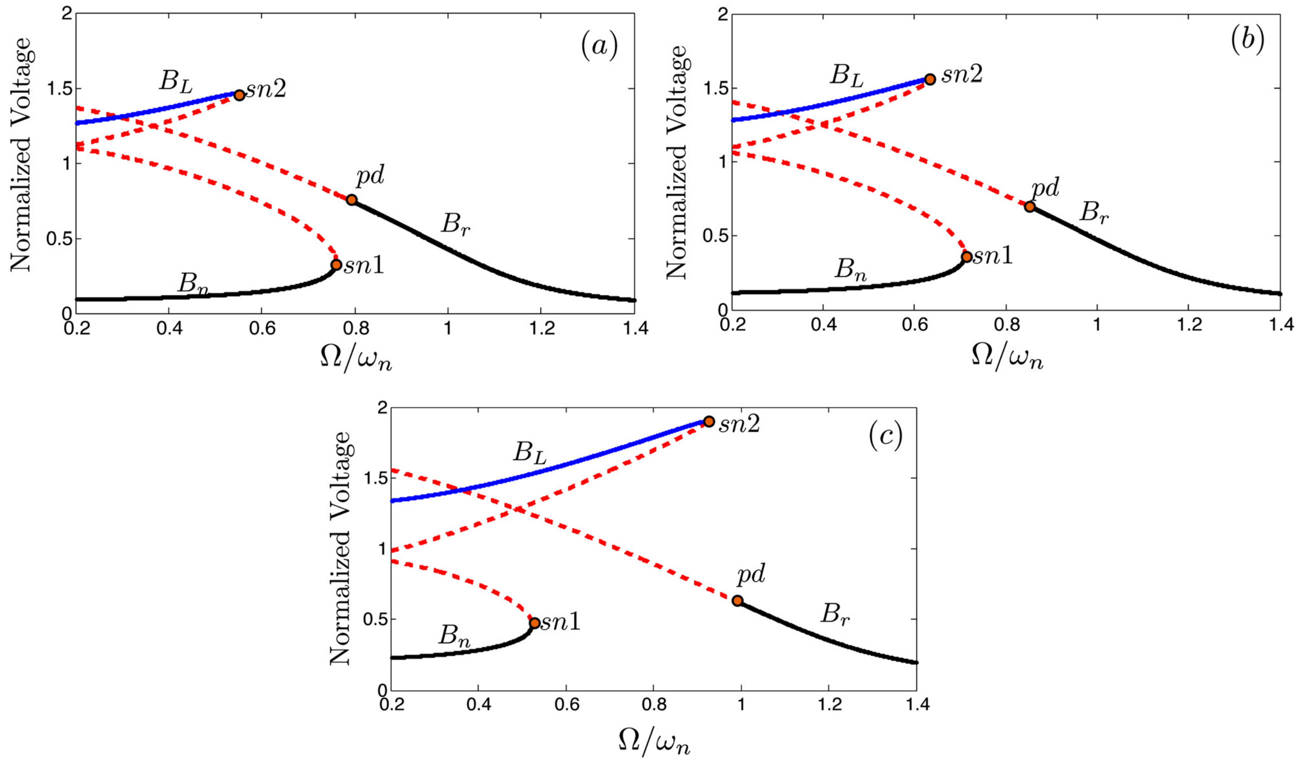


Fig. 25 Open-circuit voltage frequency response curves for a bistable harvester at three different excitation levels. Results are obtained for Eq. (4) with $\zeta = 0.05$, $r = 1.5$, $\delta = 0.5$, and a base excitation of normalized amplitude (*a*) $A = 0.08$, (*b*) $A = 0.1$, and (*c*) $A = 0.175$.

Essentially, the loci of these three points on the frequency response curve can be used as an approximate measure for the bandwidth of frequencies, wherein the harvester can produce large-amplitude voltages. When comparing Fig. 25(*a*) to Figs. 25(*b*) and 25(*c*), which are obtained for increasing values of the input excitation, it becomes evident that the period-doubling bifurcation *pd* and the saddle-node bifurcation *sn*2 occur at higher values of the frequency as the excitation magnitude is increased. On the other hand, the saddle-node bifurcation *sn*1 occurs at lower values of the frequency. This implies that the bandwidth of frequencies for which the bistable VEH can operate on the interwell branch increases with the excitation level. Such critical findings, which can only be obtained using approximate analytical solutions, should also be extended to understand the influence of the other design parameters, including the potential shape, the electromechanical coupling, and the load resistance on this bandwidth of frequencies. With this understanding, the performance of bistable VEHs can be optimized for enhanced performance.

Although bistable VEHs seemingly have certain rich and broadband dynamic response characteristics, it is not possible to fully understand their advantages without investigating their performance relative to a monostable Duffing VEH using fair comparison basis under different excitation levels. To this end, Masana and Daqaq [78] investigated the performance of their axially loaded piezoelectric VEH in both of the mono- and bistable configurations (which depends on the level of the axial load). In the bistable configuration, they considered both deep and shallow potential wells and showed that the shape of the potential function and the level of excitation play an important role in determining the relative performance of mono- and bistable VEHs in a given frequency range. In particular, they showed that the monostable VEH generally outperforms the bistable one when the level of excitation is small. Even when the excitation level is sufficiently increased to activate the interwell dynamics, a bistable VEH with deep potential wells did not provide significant enhancement in the output power as compared to the monostable device. When the bistable VEH is designed with shallow potential wells, the potential function becomes very similar to a monostable potential. Thus, in such a scenario, striking similarities were observed between both configurations in terms of their voltage and power response. Generally, their theoretical and experimental results did not clearly point towards significant improvement in the output power when the bistable VEH is utilized.

5.3 Response to Random Excitations. Conclusions based on the response of bistable VEHs to harmonic excitations cannot be extended to those exhibiting random characteristics. For random excitations, separate tools and analysis techniques borrowed from nonlinear stochastic vibrations are necessary to draw definitive conclusions about their performance in random environments. Such studies started with the work of Gammaitoni et al. [30] and Cottone et al. [75] followed by various researchers [59,88,89,98,99], including the work of Ferrari et al. [87,88], Litak et al. [89,100], and Daqaq [36,59]. Similar to monostable VEHs, research efforts can be grouped into the white and colored noise categories:

5.3.1 White Noise. Cottone et al. [75] analyzed the response of the bistable piezoelectric VEH shown in Fig. 21 to white noise excitations. They illustrated that a bistable device can provide performance improvement in the output power under white Gaussian noise only when the time constant of the harvesting circuit is very large; i.e., α is very small in Eq. (4). They explained that, since the nonlinearity is only a function of the displacement in most bistable VEHs, its maximum influence on the output power appears when the voltage is proportional to the displacement and not the velocity. This happens when α is very small in Eq. (4). By solving the FPK equation, Daqaq [36] corroborated these findings and showed that, when α is large, the mean power becomes independent of the nonlinearity and equals that obtained using an equivalent linear device. Both researchers also showed that this condition is necessary but not sufficient to guarantee enhanced performance. They demonstrated that, for a given known noise intensity, the potential well of the harvester should be intricately designed to balance the rate of interwell transitions (Kramer's rate) with wells' separation and the height of the potential barrier. Thus, it is concluded that the knowledge of the excitation intensity is essential to design a bistable VEH that can outperform an equivalent linear one under white noise. This conclusion has also been confirmed by Litak et al. [89], Halvorsen [60], and Zho and Erturk [101]. Experimentally, Masana and Daqaq [102] illustrated that a properly designed bistable harvester outperforms the monostable one unless the input excitation variance is very small, in which case both configurations yield similar levels of output voltage.

5.3.2 Colored Noise. The response of bistable VEHs to colored noise has not yet been comprehensively analyzed in the literature, mainly due to the complexities that can arise when analyzing the response of bistable systems to band-limited noise. Nevertheless, there are a few studies aiming at understanding their response behavior under such excitations. In one demonstration, Daqaq [59] studied the response of a bistable inductive generator to an exponentially correlated noise process. He obtained an approximate expression for the mean power under such excitations and showed that, for a given excitation intensity, there exists an optimal potential shape for which the power can be maximized. In another demonstration, Masana and Daqaq [102] experimentally investigated the response of a bistable VEH to band-limited excitations and studied the influence of the bandwidth and the center frequency of the excitation on the mean power as compared to a monostable VEH. They observed that, for small input accelerations, and regardless of the bandwidth of the excitation, mono- and bistable VEHs produce maximum voltage variance when the center frequency of the excitation matches the tuned oscillation frequency of the harvester, leading to the conclusion that the effect of the nonlinearity can be neglected in such conditions. As the excitation level is increased, larger voltages occurred at larger frequencies in the monostable case and at smaller frequencies in the bistable case due to the different nature of the nonlinearity in both configurations: hardening for the monostable VEH and softening within a single potential for the bistable harvester. Generally, for all acceleration levels, the bistable VEH exhibited a wider response bandwidth; that is, it is less susceptible to variations in the center frequency of the colored noise.

5.4 Apparent Issues. The potential benefits of a bistable VEH are very much contingent to its ability to operate on the large-orbit interwell periodic branch of solutions. When unique, this branch of solutions offers a wide bandwidth of frequencies for which the output power is large, making the harvester insensitive to design tolerances and excitation's frequency drifts. The main problem resides in the dependence of that bandwidth of frequencies on the excitation level and the harvester's design parameters. What follows summarizes the apparent issues with bistable VEHs.

- Excitation level: Performance of a bistable VEH is very much dependent on the excitation's level, a major issue which hinders performance of most nonlinear vibratory energy harvesters. If the excitation level is too small to activate the interwell oscillations, the dynamics remain confined to one potential well, and a bistable harvester performs similar to a monostable one. Reducing the depth of the potential wells to activate the interwell resonances renders the harvester "weakly" bistable again, approaching the behavior of a monostable VEH. Even if the excitation is large enough to allow the desired interwell oscillations, prior knowledge of the excitation amplitude is necessary to optimize the potential shape such that the VEH can yield enhanced performance.
- Nonuniqueness of solutions: As shown in Fig. 24, the large-amplitude branch favorable for energy harvesting can be accompanied by smaller-amplitude branches of solution, including aperiodic and chaotic responses. The bandwidth of frequencies where the large orbit branch of periodic solutions is unique has a complex dependence on the design parameters, including the potential shape, the electromechanical coupling, the effective damping, and, most importantly, the level of excitation. This complex dependence is not very easily resolved using numerical simulations or sets of experimental data. More analytical studies, similar to those recently proposed by Refs. [95–97], are becoming more critical to delineate this dependence and to propose techniques to possibly expand this bandwidth. Moreover, with the presence of nonunique solutions, performance metrics become quite vague, because there is no guarantee that the harvester will operate on a certain branch versus another. The probabilistic weighting approach proposed by Ref. [55], which involves using the basins of attractions as weights, showed promising results and can be extended to the bistable VEHs.
- Aperiodicity of solutions: Unlike linear and nonlinear monostable harvesters, a bistable harvester can exhibit aperiodic and chaotic responses. Generally, such signals cannot be used directly without further circuit conditioning and filtering. As such, it is still not very well-understood whether such responses are favorable or even useful for power generation. Future research efforts should focus on including more complex circuit dynamics models to evaluate the actual power output of bistable VEHs when the mechanical subsystem responds aperiodically.
- Response to random excitations: Bistability in VEHs does not seem to provide much power enhancement under white Gaussian excitations unless the time constant ratio, i.e., the ratio between the time constant of the mechanical oscillator and harvesting circuit, is very small. Indeed, it has been shown by various researchers that, when the time constant ratio is large, the nonlinearity has no measurable influence on the output power. However, such conclusions should not be arbitrarily generalized to more realistic environmental excitations, which do not possess the strict white noise characteristics [103]. In fact, it was shown experimentally that, under band-limited noise, a bistable VEH has a wider response bandwidth and is less susceptible to variations in the center frequency of the colored noise as compared to a monostable VEH [102].

6 Harvesters With Piecewise-Linear Restoring Force

In addition to the mono- and bistable VEHs with continuous nonlinear restoring forces considered earlier, the literature also contains several investigations where the restoring force is piecewise linear. The behavior of such systems is globally nonlinear, as they exhibit nonsmooth bifurcations similar to what is typically seen in the analysis of nonlinear systems. Such piecewise-defined linear restoring force can be physically realized by means of adding obstacles (stoppers) to conventional linear energy harvesters and usually result in a bilinear stiffness in its simplest case [104–109]. The resulting restoring effect might as well be piecewise-nonlinear [110], depending on the inherent stiffness characteristics of the harvester, the level of excitation, and/or the presence of nonlinear restoring force components, such as a magnetic field, as discussed in the previous sections.

In 2008, Soliman et al. [104] presented the first paper on modeling and experimental validations of an electromagnetic VEH exploiting a discontinuous linear stiffness. Along with a schematic of their cantilever-stopper arrangement, Fig. 26 shows the piecewise-linear restoring force resulting from the relative spacing of the stopper from the cantilever, which is governed by the following equation:

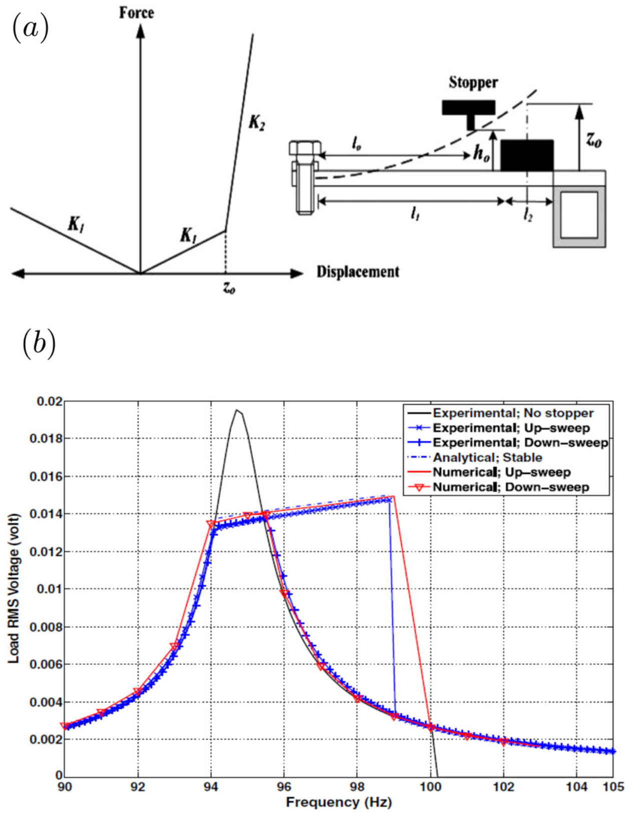


Fig. 26 (*a*) Restoring force magnitude of the piecewise linear electromagnetic harvester with one-sided stopper studied by Soliman et al. [104], and (*b*) its voltage-frequency response exhibiting bandwidth enhancement in an increasing frequency sweep. Adapted from "A Wideband Vibration-based Energy Harvester," by Soliman et al., 2008, Journal of Micromechanics and Microengineering, 18. Reproduced by permission of IOP Publishing. All rights reserved.

$$F_r(z) = \begin{cases} k_1 z, & z < z_0 \\ k_2 z + (k_1 - k_2) z_0, & z \geq z_0 \end{cases} \quad (12)$$

where the slope jumps from k_1 to k_2 as the contact between the cantilever and stopper takes place. It is clear from Fig. 26(*a*) that the slope of the restoring force, i.e., the linear stiffness, is increased when the cantilever is in contact with the stopper during the vibratory motion (i.e., $k_2 > k_1$). Therefore, it is necessary for the excitation level to be large enough, as in all previously discussed nonlinear energy harvesting concepts, such that the bilinear stiffness behavior is pronounced.

Soliman et al. [104] studied experimentally, numerically, and analytically (based on the method of averaging) the voltage-frequency response behavior of the proposed harvester. As shown in Fig. 26(*b*), the presence of stoppers extends the resonant branch of solutions to a wider range of frequencies (hardening influence) during a frequency up sweep, i.e., going from low towards high frequencies. This can be attributed to the sudden increase in the stiffness as impact occurs, which can essentially be captured via a nonlinear hardening influence. Similar to mono- and bistable Duffing VEHs, a harvester with a piecewise linear restoring force suffers from the presence of a bandwidth of frequencies where multiple stable solutions of different basins coexist, leading to hysteresis during a bidirectional frequency sweep.

It should be noted that having one-sided stopper creates lack of symmetry in the restoring force. Symmetry with respect to static equilibrium can be achieved by adding two-sided stoppers symmetrically as an alternative configuration. A downside of adding more stoppers is suggested to be the increased energy losses due to increased impacts per cycle [104]. Further procedures for design and optimization of piecewise-linear electromagnetic energy harvesters are detailed in another paper by the same research group [105].

MEMS energy harvesters exploiting impacts in electrostatic energy conversion through alternative configurations were reported by Hoffmann et al. [106] and Le et al. [107]. The former employed standard two-sided stoppers in MEMS electrostatic energy harvesting, while the MEMS architecture in the latter further exploited vibrations of the two-sided stoppers as "slave" transducers to enhance the power output. Sample frequency-response curves from these works exhibit bandwidth enhancement in up-frequency sweep, as shown in Fig. 27, which follow similar trends to the one-sided stopper case employed in macroscale electromagnetic energy harvesting [104].

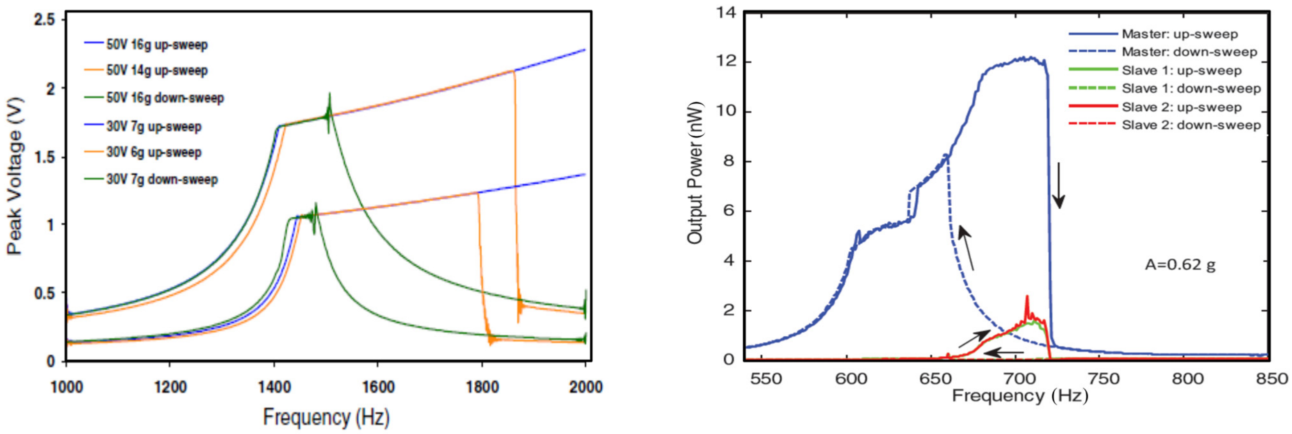
Other than these efforts on electromagnetic and electrostatic transductions, Blystad and Halvorsen [108] investigated a one-sided stopper arrangement that is similar to the configuration of Fig. 26(*a*) for bandwidth enhancement in mesoscale piezoelectric energy harvesting. More recently, MEMS piezoelectric energy harvesters employing both one-sided and two-sided stoppers were also reported by Liu et al. [109].

7 Coupling Nonlinearities

As mentioned earlier, in addition to nonlinearities that are intentionally introduced to the VEH design, nonlinearities can also be inherently present in the dynamics of a VEH, due to its geometric or transduction properties. Among such nonlinearities are those arising from coupling as well as dissipative effects that can be found in the constitutive behavior of piezoelectric and magnetostrictive materials and the electromechanical transduction of the magnetic and electrostatic potentials [28,35,39,51,111–113].

The nonlinear nature of the electromechanical coupling for a piezoelectric material has been identified in moderate-to-high voltage actuation problems for a number of years. Specifically, the nonlinear relationship between the mechanical strain and the electric field was considered by Crawley and Anderson [114], where they experimentally identified the piezoelectric coupling constant and showed that it was significantly nonlinear as a function of the material strain. duToit and Wardle [111] were among the first to suggest its influence on the performance of a VEH while considering a bimorph piezoelectric cantilever. They showed that the experimentally observed harvested power was underpredicted at resonance, as compared to a linear model in which the system parameters were identified from knowledge of the device's geometry [111]. This discrepancy was attributed to the piezoelectric coupling nonlinearity, which appeared to be of the softening type in the large-strain region.

Triplett and Quinn [28] included these piezoelectric coupling nonlinearities in a lumped-parameter model for an energy-harvesting device, together with hardening stiffness nonlinearities in the restoring force. The system was presented in an appropriate nondimensional form and subject to a single-frequency excitation to characterize the harvested power in terms of the harvester's and excitation parameters. The nondimensionalization used was able to scale the nonlinearities in the stiffness and coupling with the excitation amplitude. Based on a perturbation analysis in the absence of the coupling nonlinearity, the maximum harvested power was shown to be independent of the stiffness nonlinearity, a result confirmed earlier in Fig. 6. However, the maximum harvested power was shown to have a strong dependence on the coupling nonlinearity. Specifically, as the coupling nonlinearity was increased, the maximum harvested power increased initially up to an optimal value, beyond which it started to decrease again. A similar behavior was observed by Renno et al. [115] for the linear coupling where the drop in power was attributed to the increase in the effective damping.



(a) (b)

Fig. 27 **(*a*) Voltage and (*b*) power frequency response curves obtained via bidirectional frequency sweeps in two MEMS electrostatic energy harvesters with two-sided stoppers as reported by (*a*) Hoffmann et al. [106] and (*b*) Le et al. [107]. Part (*a*) is adapted from "Fabrication, Characterization and Modeling of Electro-static Micro-generators," by Hoffmann et al., 2009, Journal of Micromechanics and Microengineering, 19. Reproduced by permission of IOP Publishing. All rights reserved. Part (*b*) is adapted from "Microscale Electrostatic Energy Harvester Using Internal Impacts," by Le et al., 2012, Journal of Intelligent Material Systems and Structures, 23. Reproduced by permission of SAGE Publishing. All rights reserved.**

Stanton et al. [39,112] likewise considered a single-mode model for a piezoelectric cantilevered harvester and noted similar trends, in particular, with respect to the softening behavior associated with the nonlinear piezoelectric coupling. In addition, having observed mismatch in the peak amplitude of the response incorporating nonlinear constitutive behavior as compared to experimental measurements and discovering quadratic components in the electromechanical response, they included the dissipative effects in the form of quadratic damping [39,112]. Most recently, after a series of rigorous experiments and alternative mathematical frameworks of the constitutive behavior, Erturk et al. [35] considered a general constitutive equation form along with nonconservative terms and attributed the softening behavior to purely elastic terms (rather than coupling) due to low voltage levels in the energy-harvesting problem. It is worth mentioning that bending strength testing of electroded piezoelectric beams [116] of same type (and manufacturer) indicate softening in the purely mechanical stress-strain relationship in agreement with this framework [35]. Experimental results in Ref. [35] still necessitated the need for nonconservative terms as a result of mechanical and dielectric losses, which deserves further investigation and characterization.

Abdelkefi et al. [117] considered a similar system but extended the problem to a multimode model of the harvester and generalized the piezoelectric constitutive law to include a general quadratic relationship between the strain and electric field. As in Triplett and Quinn [28], the peak power and its location in the frequency spectrum was observed to be most sensitive to the nonlinearity coupling, the electric field, and the elastic strain.

In addition to the material nonlinearities in piezoelectric materials, Tvedt et al. [51] and Owens and Mann [113] considered electromechanical coupling nonlinearities in electrostatic and electromagnetic VEHs, respectively. Tvedt [51] observed that the geometric nonlinearities present in the restoring force of the suspension springs of their proposed device were more dominant than the coupling nonlinearities. Owens and Mann [113] compared the response of an electromagnetic VEH with a nonlinear electromagnetic coupling mechanism to that of a linear coupling law. It was shown that the form of the coupling that leads to the maximum harvested power depends on the system design parameters as well as the form of the excitation. Thus, they conclude that coupling nonlinearities can benefit the harvesting performance if properly incorporated into the system.

8 Nonlinearities for Low-Frequency Excitations

A major paradox currently lies in designing miniaturized linear VEHs that are capable of harnessing energy efficiently from the low-frequency excitations commonly found in nature. Specifically, since linear VEHs operate based on the basic principle of resonance, the harvester's fundamental frequency has to be equal or close to the excitation frequency for efficient energy transduction. Thus, when the size of a VEH decreases, its fundamental frequency increases, and it becomes incapable of harnessing energy efficiently from low-frequency inputs.

Since nonlinear systems exhibit superharmonic resonances that can activate large-amplitude motions at fraction integers of the fundamental frequency of the system, such resonances offer a unique and untapped opportunity for harnessing vibratory energy from excitation sources with low-frequency components. A few research efforts have investigated the prospect of utilizing superharmonic resonances of order two (half the fundamental frequency) and three (one-third the fundamental frequency) for energy harvesting. In one demonstration, Barton et al. [47] showed that the superharmonic resonances of an inductive monostable Duffing-type harvester can be used to harvest energy from low-frequency inputs. They noted that large-amplitude motions can be activated if the harvester is excited at superharmonics of order two, three, four, and five. The even harmonics, however, produced much lower amplitude motions, because the restoring force of the proposed harvester is symmetric in nature. They indicated that upconversion of the frequency results in an overall low displacement while providing a relatively higher velocity/voltage, possibly making the VEH more suitable for applications of constrained space.

In another demonstration, Masana and Daqaq [86] noticed theoretically and experimentally that the superharmonic frequency bands can activate large-amplitude motions at much lower excitation amplitudes if the VEH has a bistable potential well. Using power-frequency bifurcation maps obtained near the superharmonic resonance of order two, they showed that, for certain base acceleration levels, a bistable VEH can exhibit responses that are favorable for energy harvesting near half its fundamental frequency, $1/2\omega_n$. As shown in Fig. 28(*a*), they observed theoretically that, near half its fundamental frequency, a bistable VEH exhibits responses similar to what is seen near its fundamental frequency. These include a large branch of interwell motion, B_L, and

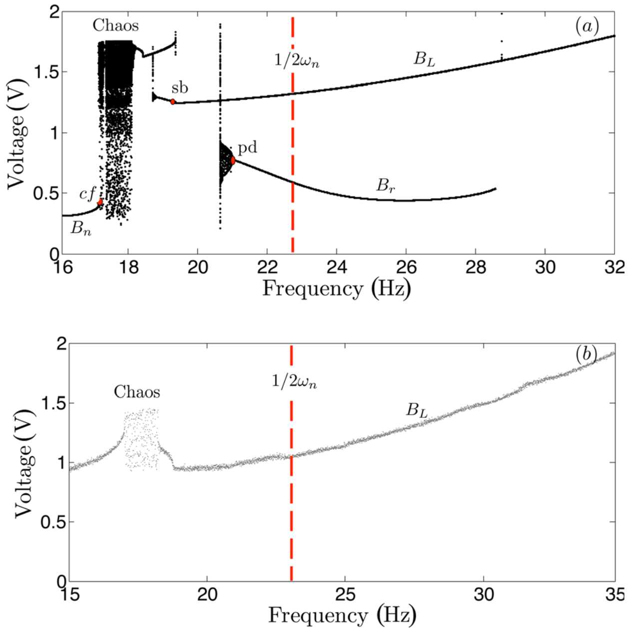


Fig. 28 Voltage-frequency response curves of a bistable VEH operating near half its fundamental frequency as reported by Masana and Daqaq [86]. (*a*) Theoretical and (*b*) experimental results.

the resonant and nonresonant branches of intrawell motion, B_r and B_n, respectively. A similar sequence of bifurcations leading to a region of chaos was also observed. In these frequency regions, the harvester was capable of producing power levels at half its fundamental frequency that are comparable to those obtained near the fundamental frequency. Quite interestingly, in the experimentally setting, the harvester was only capable of operating on the large-orbit branch of solutions, as shown in Fig. 28(*b*), further enhancing its potential.

9 Future Research Directions

We hope that this manuscript was able to draw a clear picture of current research efforts, highlighting the role and potential benefits/drawbacks of nonlinearities in energy harvesting. From the literature, it is evident that the presence of nonlinearity, intentional or inherent, has a substantial influence on the performance of energy harvesters. Whether the nonlinearity appears in the restoring force or in the coupling or it yields a mono- or bistable potential function, it obviously complicates the response behavior and, with that, the tools necessary to assess the performance of the harvester. Nonuniqueness of solutions and their aperiodicity and bifurcations that occur as parameters vary are among the important phenomena that makes developing direct performance metrics to assess the performance of nonlinear energy harvesters under different types of excitations a challenging task. Therefore, developing such metrics represents an essential first step towards understanding the role of nonlinearity in the transduction of energy harvesters.

One of the main goals of this review is to point out that the simple performance metrics currently adopted in the literature to assess performance of linear VEHs can rarely be directly extended to nonlinear harvesters. Unlike linear VEHs, the steady-state frequency response of nonlinear VEHs under a harmonic excitation may not be an accurate performance metric and, most importantly, not always a direct indication of its transduction capabilities under different types of environmental excitations. For instance, a bistable harvester that seems to have a broad steady-state bandwidth under harmonic excitations might not perform well under random or nonstationary inputs. Indeed, due to the presence of coexisting branches of stable small- and large-orbit solutions with different basins of attraction, erroneous conclusions can be drawn, if based only on a cursory inspection of the steady-state response. For nonlinear VEHs, an essential first step in the design is a careful characterization of the excitation source. Based on this analysis, separate tools and analysis techniques are necessary to draw definitive conclusions about its performance.

Perhaps contrary to one's intuition, the performance metrics of nonlinear VEHs are much easier to define when the excitation is random in nature. In such case, statistical averages of the input and output can be easily defined. Additionally, nonuniqueness and bifurcations of solutions become less of an issue because the probability density function of the harvester is always unique and independent of the response nature, even when the system is nonlinear. Indeed, the Fokker-Plank-Kolmogorov (FPK) equation governing the evolution of the probability density function of a stochastic differential equation is a linear partial differential equation, even for nonlinear systems. Thus, it has a unique solution leading to unique statistical averages. With this knowledge, the variance of the voltage and the mean power become readily available as simple performance measures. On the other hand, when the excitation is harmonic, the harvester's performance becomes a function of the excitation frequency. For linear VEHs, the output power can be easily estimated at any frequency using the linear power-frequency response curves. For nonlinear VEHs, however, the presence of coexisting branches of solution at a given frequency makes obtaining a direct measure of the output power a difficult task. Obviously, a direct averaging of the magnitude of the coexisting voltages provides an inaccurate measure of the actual performance, since each of these solutions has a different basin of attraction. The approach proposed by Quinn et al. [55], which uses the basins of attractions of the coexisting solutions as weights for averaging the output voltage, represents an important step in the right direction, because it provides a statistical measure of performance for all the physically realizable initial conditions. However, this approach can be computationally demanding, as it requires calculating the basins of attraction at each frequency within the bandwidth of interest. Furthermore, this approach completely neglects the fact that the initial conditions are not necessarily random and maybe fixed during operation. Thus, the harvester may operate on one branch of solutions at all times. Another approach could be based on providing a maximum and a minimum value for the output power at each frequency. This provides an estimate for the upper and lower performance limits of the harvester. We believe that formalizing more accurate metrics to assess the performance of nonlinear VEHs under harmonic excitations is an issue that requires immediate attention.

Another critical issue that complicates defining performance metrics for nonlinear VEHs is that the measured electric output (current or voltage) across a given load is not necessarily periodic (alternating) in nature. As such, using an rms value of the output signal can provide a false indication of actual performance. For instance, what are the implications of having a chaotic output voltage across a given load as is observed in bistable harvesters?! We believe that the answer to such a question cannot be realized unless the dynamics of more complex circuits that involve signal conditioning, filtering, and electric device modeling is included. This should constitute one of the major avenues of future research efforts.

In addition to defining effective performance metrics for nonlinear VEHs, techniques to broaden the bandwidth of frequencies where the large-orbit branches of solution are unique constitute an interesting area that requires further research. Likewise, are there nonlinear designs that extend the bandwidth for which the large-amplitude solution branches are unique? In the authors' opinion, such questions cannot be answered nor can solution techniques be developed based on numerical simulations or experiments only. Rather, such questions require the implementation of analytical approaches based on, for example, perturbation methods or global methods to better delineate the influence of the design parameters on the harvester's bandwidth. Recent approaches based on utilizing dynamic magnification by incorporating additional degrees of freedom to broaden the bandwidth of bistable energy harvesters also deserve further research attention [118].

Exploiting nonlinear interactions and internal resonances to channel energy from the lower to the higher vibration modes by designing multi–degree of freedom nonlinear VEHs with commensurate modal frequencies represents an interesting topic, which might permit harvesting energy effectively from low-frequency excitations. In particular, the concept of incorporating nonlinear mechanical attachments with low fundamental frequencies that internally resonate with a smaller nonlinear VEH of a higher fundamental frequency might permit designing scalable energy harvesters that are capable of harnessing energy from low-frequency inputs. In addition, borrowing concepts from nonlinear broadband vibration isolation might provide additional solutions for broadband and improved transduction. For example, designing nonlinear energy harvesting circuits that resonate internally with the mechanical subsystem can provide effective mechanisms to transfer energy from a mechanical oscillator to an electric device. Such concepts have been used effectively for vibration absorption and might be adapted to vibratory energy harvesting [119]. In general, previous research efforts have mainly focused on incorporating nonlinearities in the mechanical subsystem while using simple linear circuit models. However, the inclusion of more complex conditioning circuits that involve nonlinear circuit elements and battery models will open new directions that permit reaping the full benefits of the nonlinearity.

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